# Theory of Computation
# Chapter 7

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## Parameters for a Complexity Class

- model of computation: multi-string Turing machine

- modes of computation
  1. deterministic mode
  2. nondeterministic mode

- a fixed resource we wish to bound
  1. time
  2. space

- a bound $f$ mapping from $\mathbb{N}$ to $\mathbb{N}$.

# Definition 7.1: Proper Function

$f : \mathbb{N} \to \mathbb{N}$ is proper if

1. $f$ is non-decreasing (i.e., $f(n+1) \geq f(n)$);

2. there is a $k$-string TM $M_f$ with I/O such that for any input $x$ of length $n$, $M_f$ computes $\sqcup^{f(n)}$ in time $O(n + f(n))$.

# Example 7.1

1. $f(n) = c$ is proper.

2. $f(n) = \lceil \lg n \rceil$ is proper.
   Since $\lfloor \lg n \rfloor + 1$ is the length of binary digits for $n$.

3. $(\lg n)^2, n \lg n, n^2, n^3 + 3n, 2^n, \sqrt{n}, n!$ are all proper.

**Remark**  If $f$ and $g$ are proper, then so are $f + g$, $f \cdot g$, $2^g$, and $f \circ g$. $(f(n) \geq n$ for the last case)

# Definition

A Turing machine $M$ (with or without I/O, deterministic or nondeterministic) is precise if there are functions $f$ and $g$ such that for any input $x$, $M(x)$ halts after precisely $f(|x|)$ steps and uses precisely $g(|x|)$ spaces.

# Proposition 7.1

Suppose that a TM $M$ (deterministic or not) decides a language $L$ within time (or space) $f(n)$ where $f$ is a proper function. Then there is a precise TM $M'$ that decides $L$ in time (or space, resp.) $O(f(n))$.

1. Compute $M_f(x)$. Use the output of $M_f(x)$ as a "yardstick" (alarm clock).

2. Run $M$ according to the "yardstick."

# Definition: Complexity Classes

1. $\text{TIME}(f)$: deterministic time
   $\text{SPACE}(f)$: deterministic space
   $\text{NTIME}(f)$: nondeterministic time
   $\text{NSPACE}(f)$: nondeterministic space
   where $f$ is always a proper function.

2. $\text{TIME}(n^k) = \bigcup_{j>0} \text{TIME}(n^j)$ $(= \mathcal{P})$
   $\text{NTIME}(n^k) = \bigcup_{j>0} \text{NTIME}(n^j)$ $(= \mathcal{NP})$

3. $\text{PSPACE} = \text{SPACE}(n^k)$
   $\text{NPSPACE} = \text{NSPACE}(n^k)$
   $\text{EXP} = \text{TIME}(2^{n^k})$
   $\mathcal{L} = \text{SPACE}(\lg n)$
   $\mathcal{NL} = \text{NSPACE}(\lg n)$

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# Complement of Nondeterministic Classes

non-deterministic computation:

$\begin{cases} \text{accepts a string if one successful computation exists;} \\ \text{rejects a string if all computations fail.} \end{cases}$

### Definition

1. Let $L \subseteq \Sigma^*$ be a language. The complement of $L$ is $\bar{L} = \Sigma^* - L$.

2. The complement of a decision problem $A$, usually called $A$-complement, is the decision problem whose answer is "yes" if the input is "no" in $A$, "no" if the input is "yes" in $A$.

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# Example

1. SAT-complement (or coSAT): Given a Boolean expression $\phi$ in conjunctive normal form, is it unsatisfiable?

**Definition**

For any complexity class $\mathcal{C}$, let $co\mathcal{C}$ be the class $\{\bar{L}|L \in \mathcal{C}\}$.

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**Corollary** $\mathcal{C} = co\mathcal{C}$ if $\mathcal{C}$ is a deterministic time or space complexity class.

That is, all deterministic time and space complexity classes are closed under complement.

# Remark

If is an important open problem whether nondeterministic time complexity classes are closed under complement.

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# Halting Problem with Time Bounds

**Definition**

$H_f = \{M; x \mid M \text{ accepts input x after at most } f(|x|) \text{ steps}\}$
where $f(n) \geq n$ is a proper complexity function.

**Lemma 7.1**   $H_f \in \text{TIME}(f(n)^3)$. $(H_f \in \text{TIME}(f(n) \cdot \lg^2 f(n)))$

# Lemma 7.2

$H_f \notin \text{TIME}(f\lfloor \frac{n}{2} \rfloor)$.

**Proof:**   By contradiction. Suppose $M_{H_f}$ decides $H_f$ in time $f\lfloor \frac{n}{2} \rfloor$.
Define $D_f(M)$ as

$$\text{if } M_{H_f}(M; M) = \text{"yes" then "no", else "yes".}$$

What is $D_f(D_f)$?

If $D_f(D_f) =$ "yes", then $M_{H_f}(M_{D_f}; M_{D_f}) =$ "no",
"no" "yes".

Contradiction!

# The Time Hierarchy Theorem

**Theorem 7.1**
If $f(n) \geq n$ is a proper complexity function, then the class
$\mathtt{TIME}(f(n))$ is strictly contained within $\mathtt{TIME}(f(2n+1)^3)$.

**Remark**
A stronger version suggests that

$$\mathtt{TIME}(f(n)) \subsetneq \mathtt{TIME}(f(n)\lg^2 f(n)).$$

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**Corollary** $\mathcal{P}$ is a proper subset of EXP.

1. $\mathcal{P}$ is a subset of $\mathtt{TIME}(2^n)$.

2. $\mathtt{TIME}(2^n) \subsetneq \mathtt{TIME}((2^{2n+1})^3)$ (Time Hierarchy Theorem)
   $\mathtt{TIME}((2^{2n+1})^3) \subseteq \mathtt{TIME}(2^{n^2}) \subseteq$ EXP.

# The Space Hierarchy Theorem

If $f(n)$ is a proper function, then $\mathtt{SPACE}(f(n))$ is a proper subset of $\mathtt{SPACE}(f(n)\lg f(n))$.

(Note that the restriction $f(n) \geq n$ is removed from the Time Hierarchy Theorem.)

# The Gap Theorem

**Theorem 7.3**   There is a recursive function $f$ from $\mathbb{N}_0$ to $\mathbb{N}_0$ such that $\mathtt{TIME}(f(n)) = \mathtt{TIME}(2^{f(n)})$.

# The Reachability Method

**Theorem 7.4** Suppose that $f(n)$ is a proper complexity function.

1. $\text{SPACE}(f(n)) \subseteq \text{NSPACE}(f(n))$, $\text{TIME}(f(n)) \subseteq \text{NTIME}(f(n))$. ($\because$ DTM is a special NTM.)

2. $\text{NTIME}(f(n)) \subseteq \text{SPACE}(f(n))$.

3. $\text{NSPACE}(f(n)) \subseteq \text{TIME}(k^{\lg n + f(n)})$ for $k > 1$.

**Corollary**

$$\mathcal{L} \subseteq \mathcal{NL} \subseteq \mathcal{P} \subseteq \mathcal{NP} \subseteq \text{PSPACE}.$$

However, $\mathcal{L} \subsetneq \text{PSPACE}$. Hence at least one of the four inclusions is proper.

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**Theorem 7.5: (Savitch's Theorem)**
$\text{REACHABILITY} \in \text{SPACE}(\lg^2 n)$.

**Corollary**

1. $\text{NSPACE}(f(n)) \subseteq \text{SPACE}(f(n)^2)$ for any proper complexity function $f(n) \geq \lg n$.

2. $\text{PSPACE} = \text{NPSPACE}$

# Immerman-Szelepscényi Theorem

**Theorem 7.6** If $f \geq \lg n$ is a proper complexity function, then
$\texttt{NSPACE}(f(n)) = co\texttt{NSPACE}(f(n))$.

**Corollary** $\mathcal{NL} = co\mathcal{NL}$.