Theory of Computation
Chapter 9: NP-complete Problems

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NP-completeness Problems

NP: the class of languages decided by nondeterministic Turing machine in polynomial time

NP-completeness:
Cook's theorem: SAT is NP-complete.

Certificate of TM:
Hard to find an answer if there is one, but easy to verify.

SAT — a satisfying truth assignment

HAMILTON PATH — a Hamilton path

Variants of Satisfiability

- k-SAT
- 3-SAT
- 2-SAT
- MAX 2SAT
- NAESAT

k-SAT: Each clause has at most k literals.

$(\ell_1 \vee \ell_2 \vee \cdots \vee \ell_t,\ t \leq k)$

Proposition 9.2 3-SAT is NP-complete.

For any clause $C = \ell_1 \vee \ell_2 \vee \cdots \vee \ell_t$, we introduce a new variable x and split C into

$$C_1 = \ell_1 \vee \ell_2 \vee \cdots \vee \ell_{t-2} \vee x,$$

$$C_2 = \neg x \vee \ell_{t-1} \vee \ell_t.$$

Each time we obtain a clause with 3 literals. Then $F \wedge C$ is satisfiable iff $F \wedge C_1 \wedge C_2$ is satisfiable

Proposition 9.3 3-SAT remains NP-complete if each variable is restricted to appear at most three times, and each literal at most twice.

Suppose a variable x appears k times. Replace the ith x by new variable x_i for $1 \leq i \leq k$, and add

$$(\neg x_1 \vee x_2) \wedge (\neg x_2 \vee x_3) \wedge \cdots \wedge (\neg x_k \vee x_1)$$

to the expression.

$$(x_1 \Rightarrow x_2) \wedge (x_2 \Rightarrow x_3) \wedge \cdots \wedge (x_k \Rightarrow x_1)$$

$\therefore$ x_i equals x_j for $1 \leq i, j \leq k$.

Theorem 2-SAT is in NL.

Corollary 2-SAT is in P.

MAX 2SAT: Find a truth assignment that satisfies the most clauses where each clause contains at most two literals.

Theorem 9.2 MAX 2SAT is NP-complete.

Reduce 3-SAT to MAX 2SAT.

For any clause $x \vee y \vee z$ where x, y, z are literals, translate it into

$$\begin{aligned} x, y, z, w, \\ \neg x \vee \neg y, \neg y \vee \neg z, \neg z \vee \neg x, \\ x \vee \neg w, y \vee \neg w, z \vee \neg w. \end{aligned}$$

Then $x \vee y \vee z$ is satisfied iff 7 clauses are satisfied.

Let F be an instance of 3-SAT with m clauses. Then F is satisfiable iff $7m$ clauses can be satisfied in $R(F)$.

NAESAT: A clause is satisfied iff not all literals are true, and not all false. (Eg, $x \vee \neg y \vee z$, not {x=1, y=0, z=1} {x=0. y=1. z=0})

Theorem 9.3 NAESAT is NP-complete.

1. The reduction from CIRCUIT SAT to SAT;
2. Add additional new variable z to all clauses with fewer than 3 literals.

Independent set (in a graph):
$G = (V, E), I \subseteq V$. I is an independent set of G iff for all $i, j \in I$, $(i, j) \notin E$.

INDEPENDENT SET: Given a graph G and a number k, is there an independent set I of G with $|I| \geq k$?

Theorem 9.4 Independent Set is NP-complete. Reduce 3-SAT to it. If there are m clauses, let $k = m$.

1. Each clause corresponds to one triangle.

2. Complement literals are joined by an arc.

$$(x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3)$$

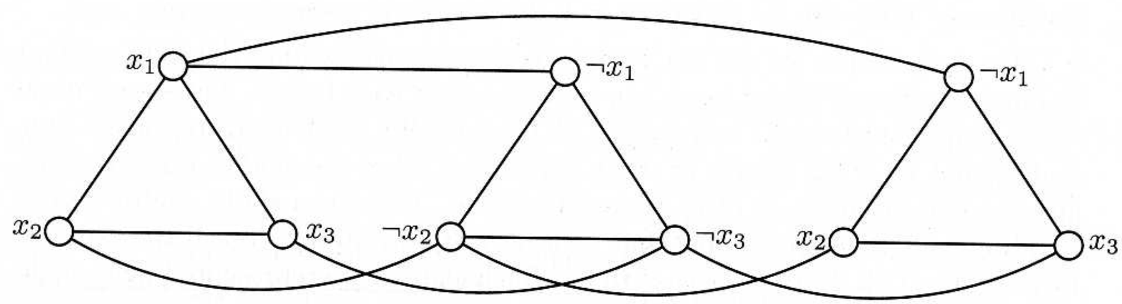


Figure 9-2. Reduction to INDEPENDENT SET.

Corollary 4-Degree Independent Set is NP-complete. (Still NP-complete when each variable appears at most 3 times and each literal appears at most twice.)

Clique: $G = (V, E)$, $C \subseteq V$. C is a clique of G iff for all $i, j \in C$, $(i, j) \in E$.

Corollary CLIQUE is NP-complete.

Node Cover: $G = (V, E)$, $N \subseteq V$ is a node cover iff for every edge $(i, j) \in E$, either $i \in N$ or $j \in N$.

Corollary NODE COVER is NP-complete.

Cut: $G = (V, E)$, $\emptyset \neq S \subsetneq V$, then $(S, V - S)$ is a cut. The size of a cut is the number of edges between S and $V - S$.

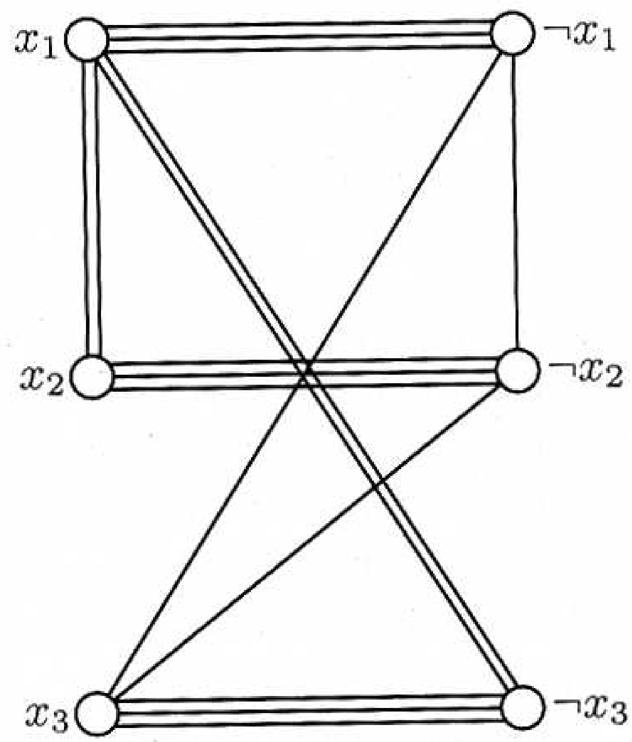


Figure 9-3. Reduction to MAX CUT.

Theorem 9.5 MAX CUT is NP-complete. Reduce NAESAT to it.

1. $F = \{C_1, C_2, \ldots, C_m\}$ clauses, each contains three literals. The variables are $x_1, x_2, \ldots, x_n$.
   $\Rightarrow G$ has $2n$ nodes, namely, $x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n$.
2. (a) For a clause $C_i = \alpha \vee \beta \vee \gamma$, add edges (α, β), (α, γ), (β, γ) into G. For a clause $C_i = \alpha \vee \alpha \vee \beta$, add (α, β), (α, β) into G.
   (b) For any variable x_i, let n_i be the number of occurrences of either x_i or $\neg x_i$ (i.e., their sums). Add n_i edges between x_i and $\neg x_i$. (3m edges are added in total.)
3. If F is NAESAT, let S be the set of literals that is true. Then $(S, V - S)$ is a cut of size

$$2m + 3m = 5m.$$

4. If G has a cut S of size $5m$ or more, without loss of generality, we assume x_i and $\neg x_i$ are in different side. There are exactly $3m$ edges introduced in 2.(b). There are at most $2m$ edges introduced in 2.(a), which equals to $2m$ if and only if all clauses are NAESAT.

Max Bisection: A special MAX CUT with $|S| = |V - S|$.

Lemma 9.1 MAX BISECTION is NP-complete.

Indeed, the proof of Theorem 9.5 is a one. Or, simply add $|V|$ isolated nodes into G.

Bisection Width: Separate the nodes into two equal parts with minimum cut.

Remark It is a generalization of MIN CUT, which is in P. (MAX FLOW=MIN CUT).

Theorem 9.6 BISECTION WIDTH is NP-complete.

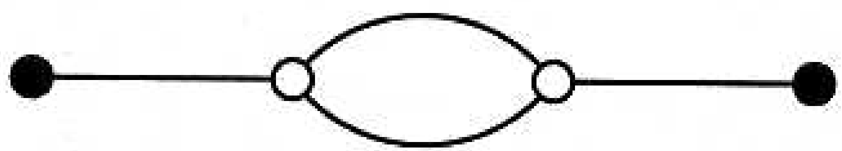
Let $G = (V, E)$ where $|V| = 2n$, then G has a bisection of size k if and only if the complement of G has a bisection of size $n^2 - k$.

Hamilton Path: Given an undirected graph G, does it have a Hamilton path?

Theorem HAMILTON PATH is NP-complete.

Reduce 3-SAT to it.

1. choice gadget



2. consistency gadget

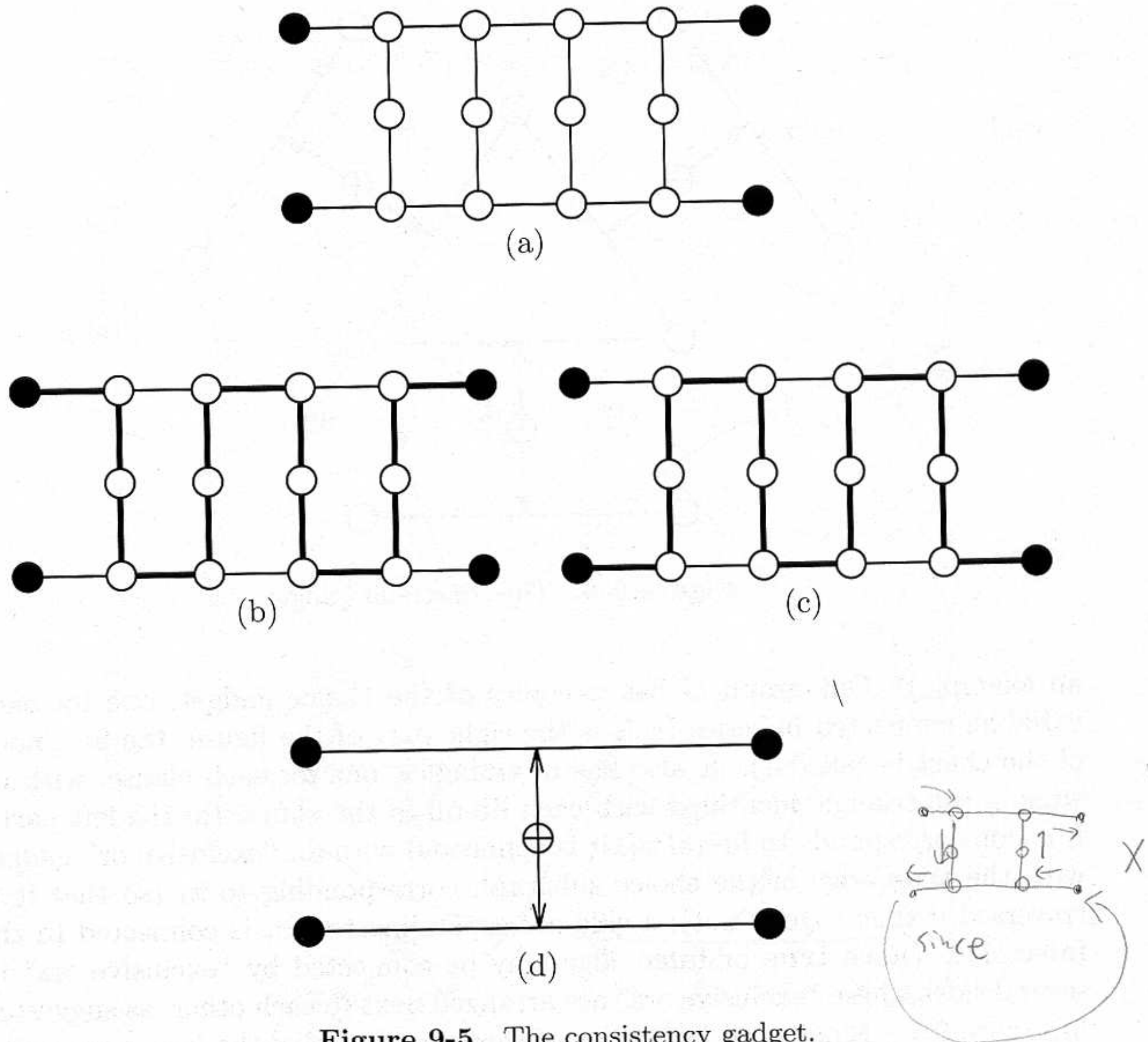


Figure 9-5. The consistency gadget.

3. constraint gadget

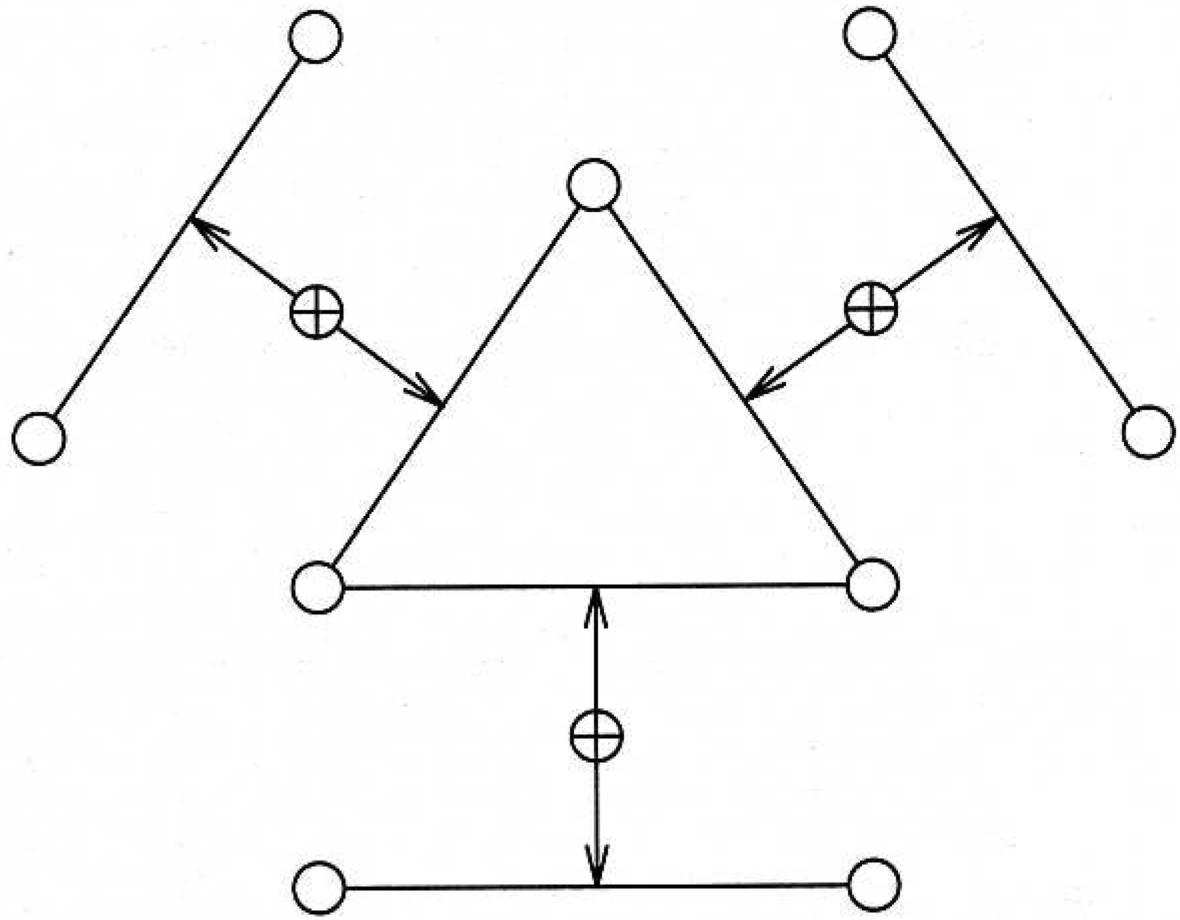
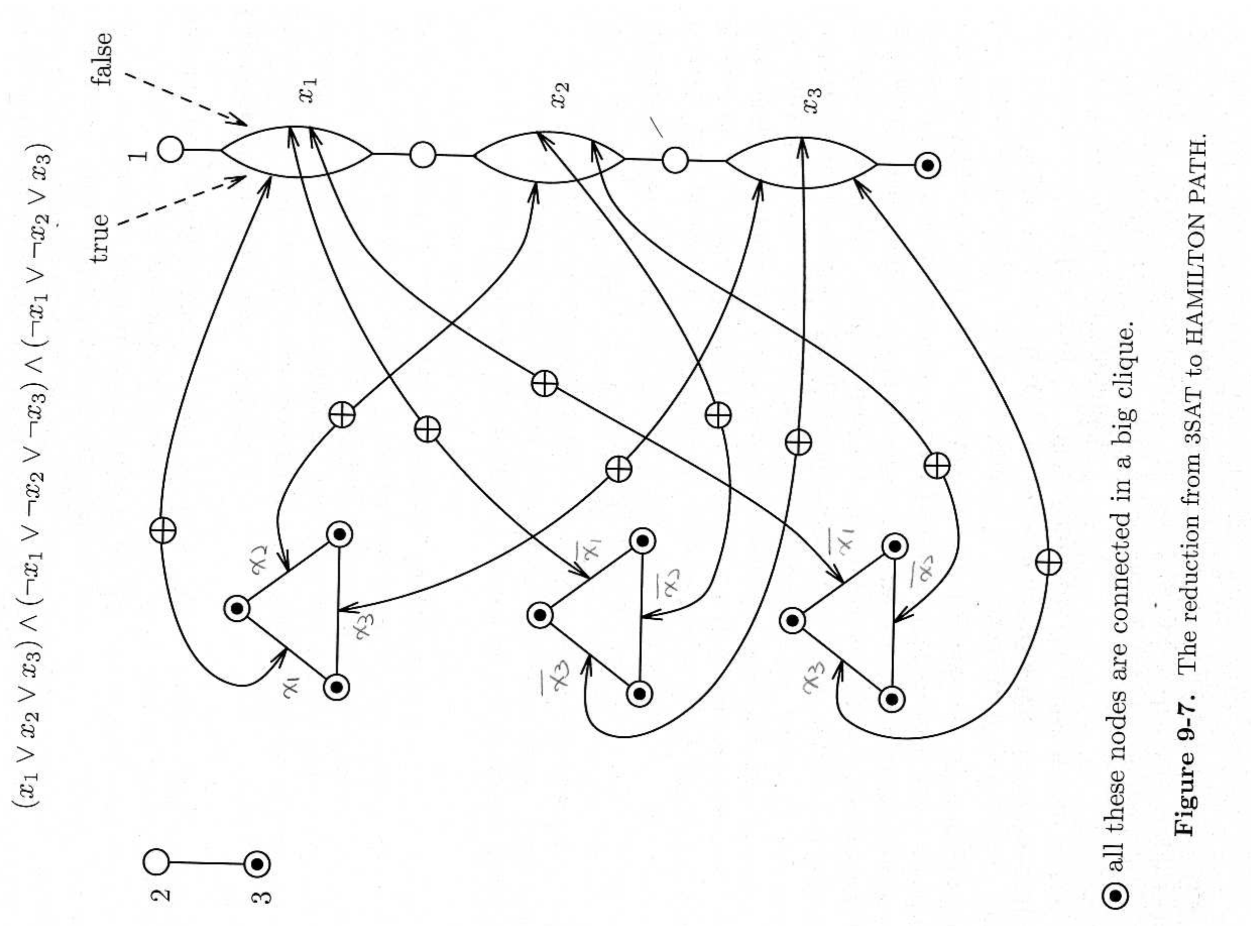


Figure 9-6. The constraint gadget.

4. Reduction from 3-SAT to HAMILTON PATH:

(a) Start from node 1, end with node 2.

(b) All $\odot$ nodes are connedted in a big clique.



Corollary TSP(D) is NP-complete.

Reduce HAMILTON PATH to it.

$$d(i,j) = \begin{cases} 1 & \text{if } (i,j) \text{ is an edge in } G; \\ 2 & \text{otherwise.} \end{cases}$$

We also add an extra node that connects to other nodes with distance 1.
G has an HP iff $R(G)$ has an HC of length $n+1$.

k-coloring of a graph: Color a graph with at most k colors such that no two adjacent nodes have the same color.

Theorem 9.8 3-Coloring is NP-complete.

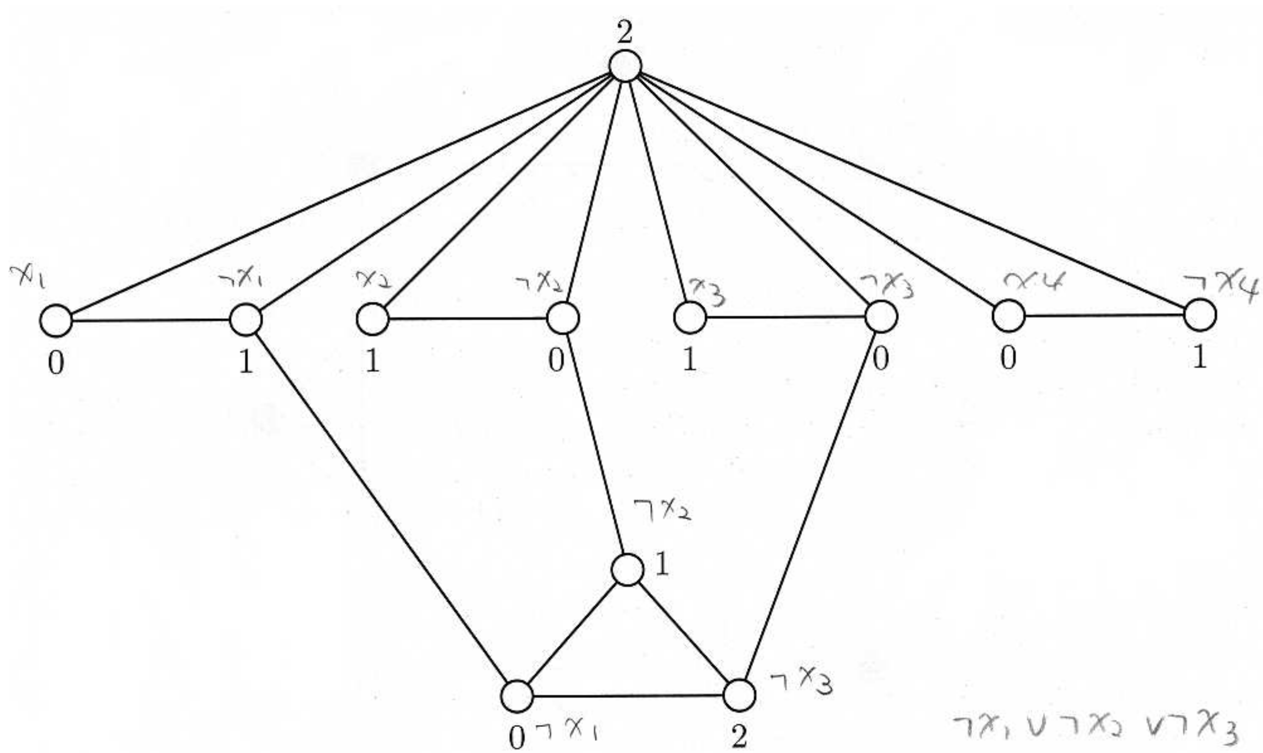


Figure 9-8. The reduction to 3-COLORING.

Reduce NAESAT to it.

1. choice gadget: upper part

2. constraint gadget: lower part

Tripartite Matching: Given $T \subseteq B \times G \times H$, $|B| = |G| = |H| = n$, try to find n triples in T s.t. no two of which have a component in common.
(B: boys, G: girls, H: homes)

Theorem 9.8 TRIPARTITE MATCHING is NP-complete.

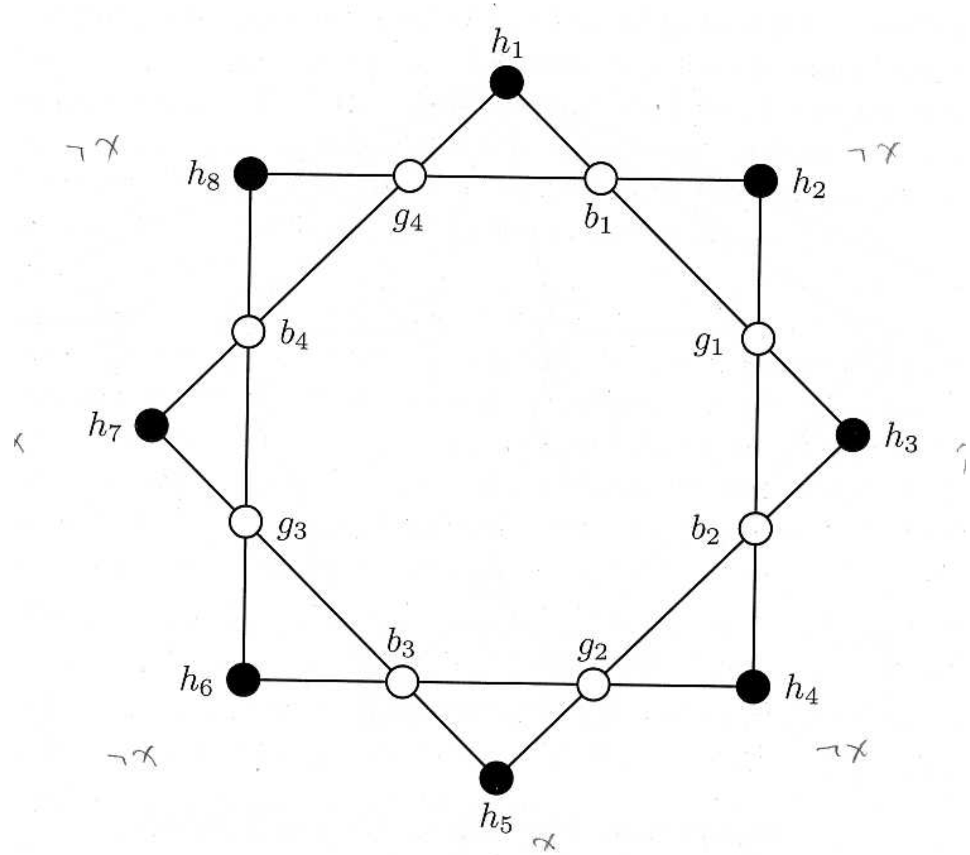


Figure 9-9. The choice-consistency gadget.

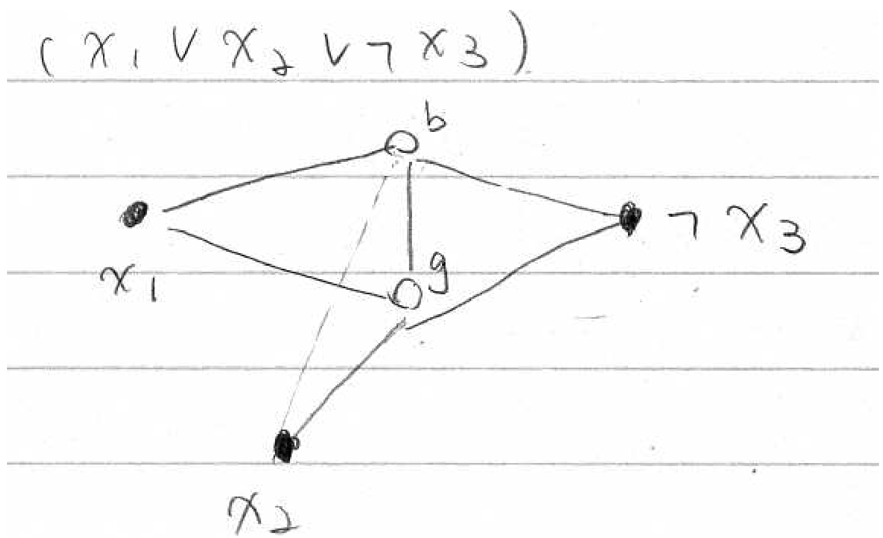
Reduce 3-SAT to it.

1. For each variable x_i, we construct a choice-consistency gadget.
   (a) Let k be the maximum of the occurrences of x and $\neg x$ (i.e.,

$\max\{\mathrm{occ}(x_i), \mathrm{occ}(\neg x_i)\}$).

(b) There are k boys, k girls, $2k$ homes in this gadget.

2. For each clause $(\alpha \vee \beta \vee \gamma)$, construct a new added triple (b, g, h) where h is either α, β, or γ, not joined by another triple in this step.



3. Suppose there are m clauses. Since $\mathrm{occ}(x_i) + \mathrm{occ}(\neg x_i) \leq 2k_i$, we have $3m \leq |H|$. Hence, there are at least $3m$ homes. The number of boys is $\frac{|H|}{2} + m \leq |H|$. Introduce ℓ more boys &

girls such that $|B| = |G| = |H|$. For each of the ℓ boys and girls, add $|H|$ triples that connect to all homes.

Set Covering: $F = \{S_1, \ldots, S_m\}$ of subsets of a finite set U. Find a minimum sets in F whose union is U.

Set Packing: $F = \{S_1, \ldots, S_m\}$ of subsets of a finite set U. Find a maximum sets in F that are pairwise disjoint.

Exact Cover by 3-Set: $F = \{S_1, \ldots, S_n\}$ of subsets of a finite set U, and $|S_i| = 3$, $|U| = 3m$ for some $m \leq n$. Find m sets in F that are disjoint and have U as their union.

All of these problems are generalization of TRIPARTITE MATCHING. Hence, they are all NP-complete.

SET COVERING SET PACKING

generalize

EXACT COVER BY 3-SET

specialize

TRIPARTITE MATCHING

Integer Programming: Given a system of linear inequalities with integer coefficients, does it have an integer solution?

Theorem INTEGER PROGRAMMING is NP-complete.

Reduce SET COVERING to it. Let $F = \{S_1, \ldots, S_n\}$ be subsets of U. $x = (x_1\ x_2 \cdots x_n)^t$. $x_i = \begin{cases} 1 & \text{if } S_i \text{ is in the cover;} \\ 0 & \text{otherwise.} \end{cases}$

$A = (a_{i,j})$, $a_{i,j} = 1$ iff the ith element in U belongs to S_j.

$$\Rightarrow \begin{cases} Ax \geq \vec{1}; \\ \sum_{i=1}^{n} x_i \leq B, \text{where } B \text{ is the budget;} \\ 0 \leq x_i \leq 1. \end{cases}$$

Knapsack: $\{1, 2, \ldots, n\}$, n items. Item i has value $v_i > 0$ and weight $w_i > 0$. Try to find a subset $S \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i \geq K$ for some W and K.

Theorem 9.10 KNAPSACK is NP-complete.

$\to$	0	0	0	1	0	1	1	0	0	0	0	0
	1	1	0	0	1	0	0	0	0	0	0	0
$\to$	1	0	1	0	0	0	0	1	0	0	0	0
$\to$	0	1	0	0	0	0	0	0	1	0	0	1
	0	0	1	1	1	0	0	0	0	0	0	0
$\to$	0	0	0	0	1	0	0	0	0	1	1	0
$+$	1	0	1	1	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	1	1	1	1	1

Figure 9.10. Reduction to KNAPSACK.

Reduce Exact Cover By 3-Set to it. $\{S_1, S_2, \ldots, S_n\}$, an instance of Exact Cover By 3-Set, $U = \{1, 2, \ldots, 3m\}$.
Let $v_i = w_i = \sum_{j \in S_i} (n+1)^{3m-j}$ and $W = K = \sum_{j=0}^{3m-1} (n+1)^j$.
(Never carry.)

Proposition 9.4 Any instance of KNAPSACK can be solved in $O(nW)$ time, where n is the number of items and W is the weight limit.

We can solve this by dynamic programming.
$V(w, i)$: the largest value attainable by selecting some among the first i items so that the total weight is no more than w.

$$\begin{cases} V(w, i+1) = \max\{V(w, i), [w \geq w_{i+1}](v_{i+1} + V(w - w_{i+1}, i))\} \\ \quad \text{for } i \geq 0 \text{ and } 0 \leq w \leq W; \\ V(w, 0) = 0 \quad \text{for } 0 \leq w \leq W. \end{cases}$$

If $V(W, n) \geq K$, then answer "yes."