Theory of Computation
Chapter 7: Relations Between Complexity Classes

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Apr. 14, 2003
Mar. 1, 2009

Parameters for a Complexity Class

- model of computation: multi-string Turing machine
- modes of computation
  1. deterministic mode
  2. nondeterministic mode
- a resource we wish to bound
  1. time
  2. space
- a bound f mapping from $\mathbb{N}$ to $\mathbb{N}$.

Definition 7.1: Proper Function

$f : \mathbb{N} \rightarrow \mathbb{N}$ is proper if

1. f is non-decreasing (i.e., $f(n+1) \geq f(n)$);
2. there is a k-string TM M_f with I/O such that for any input x of length n, M_f computes $\sqcup^{f(n)}$ in time $O(n + f(n))$.

Example 7.1

1. $f(n) = c$ is proper.
2. $f(n) = \lceil \lg n \rceil$ is proper.
   Since $\lfloor \lg n \rfloor + 1$ is the length of binary digits for n.
3. $(\lg n)^2, n \lg n, n^2, n^3 + 3n, 2^n, \sqrt{n}, n!$ are all proper.

Remark If f and g are proper, then so are $f + g$, $f \cdot g$, 2^g, and $f \circ g$. ($f(n) \geq n$ for the last case)

Definition

A Turing machine M (with or without I/O, deterministic or nondeterministic) is precise if there are functions f and g such that for any input x, $M(x)$ halts after precisely $f(|x|)$ steps and uses precisely $g(|x|)$ spaces.

Proposition 7.1

Suppose that a TM M (deterministic or not) decides a language L within time (or space) $f(n)$ where f is a proper function. Then there is a precise TM M' that decides L in time (or space, resp.) $O(f(n))$.

1. Compute $M_f(x)$. Use the output of $M_f(x)$ as a "yardstick" (alarm clock).
2. Run M according to the "yardstick."

Definition: Complexity Classes

1. $\texttt{TIME}(f)$: deterministic time

   $\texttt{SPACE}(f)$: deterministic space

   $\texttt{NTIME}(f)$: nondeterministic time

   $\texttt{NSPACE}(f)$: nondeterministic space

   where f is always a proper function.

2. $\texttt{TIME}(n^k) = \bigcup_{j>0} \texttt{TIME}(n^j)$ $(= \mathcal{P})$

   $\texttt{NTIME}(n^k) = \bigcup_{j>0} \texttt{NTIME}(n^j)$ $(= \mathcal{NP})$

3. $\text{PSPACE} = \texttt{SPACE}(n^k)$

   $\text{NPSPACE} = \texttt{NSPACE}(n^k)$

   $\text{EXP} = \texttt{TIME}(2^{n^k})$

   $\mathcal{L} = \texttt{SPACE}(\lg n)$

   $\mathcal{NL} = \texttt{NSPACE}(\lg n)$

Complement of a Decision Problem

1. A decision problem L is a triple (M_F, I_P, I_N) where
   (a) $I_P \cup I_N \subseteq \Sigma^*$ and $I_P \cap I_N = \emptyset$;
   (b) M_F is a Turing machine such that $M_F(x) =$ "yes" when $x \in I_P \cup I_N$ and $M_F(x) =$ "no" when $x \notin I_P \cup I_N$.

   We call $x \in \Sigma^*$ a positive instance whenever $x \in I_P$ and $x \in \Sigma^*$ a negative instance whenever $x \in I_N$. The purpose of M_F is to verify the format of instances for L.

2. A decision problem is called decidable when there exists a Turing machine D such that $D(x) =$ "yes" for $x \in I_P$ and $D(x) =$ "no" for $x \in I_N$.

3. $\bar{L} = (M'_F, I'_P, I'_N)$ is called the complement of L when $M'_F = M_F$, $I'_P = I_N$, and $I'_N = I_P$. That is, they swap positive and negative instances.

Complement of Complexity Classes

Definition
For any complexity class $\mathcal{C}$, let $co\mathcal{C}$ be the class $\{\bar{L} | L \in \mathcal{C}\}$.

Corollary $\mathcal{C} = co\mathcal{C}$ if $\mathcal{C}$ is a deterministic time or space complexity class.
That is, all deterministic time and space complexity classes are closed under complement.

Complement of Nondeterministic Classes

Non-deterministic computation N_1 (for L):

$$\begin{cases} \text{accepts a string } x \text{ if one successful computation exists; (for } x \in I_P) \\ \text{rejects a string } x \text{ if all computations fail. (for } x \in I_N) \end{cases}$$

Non-deterministic computation N_2 (for $\bar{L}$):

$$\begin{cases} \text{accepts a string } x \text{ if one successful computation exists; (for } x \in I'_P) \\ \text{rejects a string } x \text{ if all computations fail. (for } x \in I'_N) \end{cases}$$

where $I'_P = I_N$ and $I'_N = I_P$.

We cannot obtain N_2 from N_1 by simply interchanging the "yes"/"no" answer!

Example

1. SAT-complement (or coSAT): Given a Boolean expression ϕ in conjunctive normal form, is it unsatisfiable?

Remark

It is an important open problem whether nondeterministic time complexity classes are closed under complement.

Halting Problem with Time Bounds

Definition

$H_f = \{M; x|\ M$ accepts input x after at most $f(|x|)$ steps$\}$
where $f(n) \geq n$ is a proper complexity function.

Lemma 7.1 $H_f \in \texttt{TIME}(f(n)^3)$ where $n = |M; x|$.
$(H_f \in \texttt{TIME}(f(n) \cdot \lg^2 f(n)))$

Lemma 7.2

$H_f \notin \texttt{TIME}(f(\lfloor \frac{n}{2} \rfloor))$.

Proof: By contradiction. Suppose M_{H_f} decides H_f in time $f(\lfloor \frac{n}{2} \rfloor)$. Define $D_f(M)$ as

if $M_{H_f}(M; M)$= "yes" then "no", else "yes".

What is $D_f(D_f)$?

If $D_f(D_f)$= "yes", then $M_{H_f}(M_{D_f}; M_{D_f})$ = "no",
"no" "yes".

Contradiction!

The Time Hierarchy Theorem

Theorem 7.1

If $f(n) \geq n$ is a proper complexity function, then the class $\texttt{TIME}(f(n))$ is strictly contained within $\texttt{TIME}(f(2n+1)^3)$.

Remark

A stronger version suggests that

$$\texttt{TIME}(f(n)) \subsetneq \texttt{TIME}(f(n) \lg^2 f(n)).$$

Corollary $\mathcal{P}$ is a proper subset of EXP.

1. $\mathcal{P}$ is a subset of $\texttt{TIME}(2^n)$.
2. $\texttt{TIME}(2^n) \subsetneq \texttt{TIME}((2^{2n+1})^3)$ (Time Hierarchy Theorem)
   $\texttt{TIME}((2^{2n+1})^3) \subseteq \texttt{TIME}(2^{n^2}) \subseteq \text{EXP}$.

The Space Hierarchy Theorem

If $f(n)$ is a proper function, then $\texttt{SPACE}(f(n))$ is a proper subset of $\texttt{SPACE}(f(n) \lg f(n))$.

(Note that the restriction $f(n) \geq n$ is removed from the Time Hierarchy Theorem.)

The Gap Theorem

Theorem 7.3 There is a recursive function f from $\mathbb{N}_0$ to $\mathbb{N}_0$ such that $\mathtt{TIME}(f(n)) = \mathtt{TIME}(2^{f(n)})$.

The Reachability Method

Theorem 7.4 Suppose that $f(n)$ is a proper complexity function.

1. $\texttt{SPACE}(f(n)) \subseteq \texttt{NSPACE}(f(n))$, $\texttt{TIME}(f(n)) \subseteq \texttt{NTIME}(f(n))$. ($\because$ DTM is a special NTM.)
2. $\texttt{NTIME}(f(n)) \subseteq \texttt{SPACE}(f(n))$.
3. $\texttt{NSPACE}(f(n)) \subseteq \texttt{TIME}(k^{\lg n + f(n)})$ for $k > 1$.

Corollary

$$\mathcal{L} \subseteq \mathcal{NL} \subseteq \mathcal{P} \subseteq \mathcal{NP} \subseteq \text{PSPACE}.$$

However, $\mathcal{L} \subsetneq$ PSPACE. Hence at least one of the four inclusions is proper. (Space Hierarchy Theorem)

Theorem 7.5: (Savitch's Theorem)
REACHABILITY $\in$ $\texttt{SPACE}(\lg^2 n)$.

Corollary

1. $\texttt{NSPACE}(f(n)) \subseteq \texttt{SPACE}(f(n)^2)$ for any proper complexity function $f(n) \geq \lg n$.
2. PSPACE = NPSPACE

Immerman-Szelepscényi Theorem

Theorem 7.6 If $f \geq \lg n$ is a proper complexity function, then $\texttt{NSPACE}(f(n)) = co\texttt{NSPACE}(f(n))$.

Corollary $\mathcal{NL} = co\mathcal{NL}$.