

Theory of Computation

Chapter 7

Guan-Shieng Huang

Apr. 14, 2003

Feb. 19, 2006

Parameters for a Complexity Class

- model of computation: multi-string Turing machine
- modes of computation
 1. deterministic mode
 2. nondeterministic mode
- a fixed resource we wish to bound
 1. time
 2. space
- a bound f mapping from \mathbb{N} to \mathbb{N} .

Definition 7.1: Proper Function

$f : \mathbb{N} \rightarrow \mathbb{N}$ is proper if

1. f is non-decreasing (i.e., $f(n + 1) \geq f(n)$);
2. there is a k -string TM M_f with I/O such that for any input x of length n , M_f computes $\sqcup^{f(n)}$ in time $O(n + f(n))$.

Example 7.1

1. $f(n) = c$ is proper.
2. $f(n) = \lceil \lg n \rceil$ is proper.
Since $\lfloor \lg n \rfloor + 1$ is the length of binary digits for n .
3. $(\lg n)^2, n \lg n, n^2, n^3 + 3n, 2^n, \sqrt{n}, n!$ are all proper.

Remark If f and g are proper, then so are $f + g$, $f \cdot g$, 2^g , and $f \circ g$. ($f(n) \geq n$ for the last case)

Definition

A Turing machine M (with or without I/O, deterministic or nondeterministic) is precise if there are functions f and g such that for any input x , $M(x)$ halts after precisely $f(|x|)$ steps and uses precisely $g(|x|)$ spaces.

Proposition 7.1

Suppose that a TM M (deterministic or not) decides a language L within time (or space) $f(n)$ where f is a proper function. Then there is a precise TM M' that decides L in time (or space, resp.) $O(f(n))$.

1. Compute $M_f(x)$. Use the output of $M_f(x)$ as a “yardstick” (alarm clock).
2. Run M according to the “yardstick.”

Definition: Complexity Classes

1. $\text{TIME}(f)$: deterministic time
 $\text{SPACE}(f)$: deterministic space
 $\text{NTIME}(f)$: nondeterministic time
 $\text{NSPACE}(f)$: nondeterministic space
where f is always a **proper function**.
2. $\text{TIME}(n^k) = \bigcup_{j>0} \text{TIME}(n^j)$ ($= \mathcal{P}$)
 $\text{NTIME}(n^k) = \bigcup_{j>0} \text{NTIME}(n^j)$ ($= \mathcal{NP}$)
3. $\text{PSPACE} = \text{SPACE}(n^k)$
 $\text{NPSPACE} = \text{NSPACE}(n^k)$
 $\text{EXP} = \text{TIME}(2^{n^k})$
 $\mathcal{L} = \text{SPACE}(\lg n)$
 $\mathcal{NL} = \text{NSPACE}(\lg n)$

Complement of Nondeterministic Classes

non-deterministic computation:

- { accepts a string if one successful computation exists;
- { rejects a string if all computations fail.

Definition

1. Let $L \subseteq \Sigma^*$ be a language. The complement of L is $\bar{L} = \Sigma^* - L$.
2. The complement of a decision problem A , usually called A -complement, is the decision problem whose answer is “yes” if the input is not in A , “no” if the input is in A .

Example

1. SAT-complement (or coSAT): Given a Boolean expression ϕ in conjunctive normal form, is it **unsatisfiable**?

Definition

For any complexity class \mathcal{C} , let $co\mathcal{C}$ be the class $\{\bar{L} \mid L \in \mathcal{C}\}$.

Corollary $\mathcal{C} = co\mathcal{C}$ if \mathcal{C} is a deterministic time or space complexity class.

That is, all deterministic time and space complexity classes are closed under complement.

Remark

It is an important [open problem](#) whether nondeterministic time complexity classes are closed under complement.

Halting Problem with Time Bounds

Definition

$H_f = \{M; x \mid M \text{ accepts input } x \text{ after at most } f(|x|) \text{ steps}\}$
where $f(n) \geq n$ is a proper complexity function.

Lemma 7.1 $H_f \in \text{TIME}(f(n)^3)$ where $n = |M; x|$.
($H_f \in \text{TIME}(f(n) \cdot \lg^2 f(n))$)

Lemma 7.2

$H_f \notin \text{TIME}(f(\lfloor \frac{n}{2} \rfloor))$.

Proof: By contradiction. Suppose M_{H_f} decides H_f in time $f(\lfloor \frac{n}{2} \rfloor)$. Define $D_f(M)$ as

if $M_{H_f}(M; M) = \text{“yes”}$ then “no”, else “yes”.

What is $D_f(D_f)$?

If $D_f(D_f) = \text{“yes”}$, then $M_{H_f}(M_{D_f}; M_{D_f}) = \text{“no”}$,
“no” “yes”.

Contradiction!

The Time Hierarchy Theorem

Theorem 7.1

If $f(n) \geq n$ is a proper complexity function, then the class $\text{TIME}(f(n))$ is strictly contained within $\text{TIME}(f(2n + 1)^3)$.

Remark

A stronger version suggests that

$$\text{TIME}(f(n)) \subsetneq \text{TIME}(f(n) \lg^2 f(n)).$$

Corollary \mathcal{P} is a proper subset of EXP.

1. \mathcal{P} is a subset of $\text{TIME}(2^n)$.
2. $\text{TIME}(2^n) \subsetneq \text{TIME}((2^{2n+1})^3)$ (Time Hierarchy Theorem)
 $\text{TIME}((2^{2n+1})^3) \subseteq \text{TIME}(2^{n^2}) \subseteq \text{EXP}$.

The Space Hierarchy Theorem

If $f(n)$ is a proper function, then $\text{SPACE}(f(n))$ is a proper subset of $\text{SPACE}(f(n) \lg f(n))$.

(Note that the restriction $f(n) \geq n$ is removed from the Time Hierarchy Theorem.)

The Gap Theorem

Theorem 7.3 There is a recursive function f from \mathbb{N}_0 to \mathbb{N}_0 such that $\text{TIME}(f(n)) = \text{TIME}(2^{f(n)})$.

The Reachability Method

Theorem 7.4 Suppose that $f(n)$ is a proper complexity function.

1. $\text{SPACE}(f(n)) \subseteq \text{NSPACE}(f(n))$, $\text{TIME}(f(n)) \subseteq \text{NTIME}(f(n))$. (\because DTM is a special NTM.)
2. $\text{NTIME}(f(n)) \subseteq \text{SPACE}(f(n))$.
3. $\text{NSPACE}(f(n)) \subseteq \text{TIME}(k^{\lg n + f(n)})$ for $k > 1$.

Corollary

$$\mathcal{L} \subseteq \mathcal{NL} \subseteq \mathcal{P} \subseteq \mathcal{NP} \subseteq \text{PSPACE}.$$

However, $\mathcal{L} \subsetneq \text{PSPACE}$. Hence at least one of the four inclusions is proper. (Space Hierarchy Theorem)

Theorem 7.5: (Savitch's Theorem)

$\text{REACHABILITY} \in \text{SPACE}(\lg^2 n)$.

Corollary

1. $\text{NSPACE}(f(n)) \subseteq \text{SPACE}(f(n)^2)$ for any proper complexity function $f(n) \geq \lg n$.
2. $\text{PSPACE} = \text{NPSPACE}$

Immerman-Szelepcényi Theorem

Theorem 7.6 If $f \geq \lg n$ is a proper complexity function, then $\text{NSPACE}(f(n)) = \text{coNSPACE}(f(n))$.

Corollary $\mathcal{NL} = \text{coNL}$.