Complexity Classes

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Parameters for a Complexity Class

- model of computation: multi-string Turing machine
- modes of computation
  1. deterministic mode
  2. nondeterministic mode
- a resource we wish to bound
  1. time
  2. space
- a bound f mapping from $\mathbb{N}$ to $\mathbb{N}$.

Definition 7.1: Proper Function

$f : \mathbb{N} \rightarrow \mathbb{N}$ is proper if

1. f is non-decreasing (i.e., $f(n+1) \geq f(n)$);
2. there is a k-string TM M_f with I/O such that for any input x of length n, M_f computes $\sqcup^{f(n)}$ in time $O(n + f(n))$.

Definition: Complexity Classes

1. $\texttt{TIME}(f)$: deterministic time

   $\texttt{SPACE}(f)$: deterministic space

   $\texttt{NTIME}(f)$: nondeterministic time

   $\texttt{NSPACE}(f)$: nondeterministic space

   where f is always a proper function.

2. $\texttt{TIME}(n^k) = \bigcup_{j>0} \texttt{TIME}(n^j)$ $(= \mathcal{P})$

   $\texttt{NTIME}(n^k) = \bigcup_{j>0} \texttt{NTIME}(n^j)$ $(= \mathcal{NP})$

3. $\text{PSPACE} = \texttt{SPACE}(n^k)$

   $\text{NPSPACE} = \texttt{NSPACE}(n^k)$

   $\text{EXP} = \texttt{TIME}(2^{n^k})$

   $\mathcal{L} = \texttt{SPACE}(\lg n)$

   $\mathcal{NL} = \texttt{NSPACE}(\lg n)$

Complement of a Decision Problem

Definition

1. Let $L \subseteq \Sigma^*$ be a language. The complement of L is $\bar{L} = \Sigma^* - L$.
2. However, we often consider languages with certain format, i.e. the set of all graphs with degree ≤ 4. In this case, we remove instances whose formats are not legal.
3. The complement of a decision problem A, usually called A-complement, is the decision problem whose answer is "yes" if the input is not in A, "no" if the input is in A.

Complement of Complexity Classes

Definition

For any complexity class $\mathcal{C}$, let $co\mathcal{C}$ be the class $\{L|\ \bar{L} \in \mathcal{C}\}$.

Corollary $\mathcal{C} = co\mathcal{C}$ if $\mathcal{C}$ is a deterministic time or space complexity class.

That is, all deterministic time and space complexity classes are closed under complement since we can simply exchange its "yes"/"no" answer.

Complement of Nondeterministic Classes

non-deterministic computation:

$$\begin{cases} \text{accepts a string if one successful computation exists;} \\ \text{rejects a string if all computations fail.} \end{cases}$$

Example

1. SAT-complement (or coSAT): Given a Boolean expression ϕ in conjunctive normal form, is it unsatisfiable?

However, we can not simply exchange the "yes"/"no" answer of a non-deterministic Turing machine for this purpose.

Remark

It is an important open problem whether nondeterministic time complexity classes are closed under complement.

Halting Problem with Time Bounds

Definition
$H_f = \{M; x| \ M$ accepts input x after at most $f(|x|)$ steps$\}$
where $f(n) \geq n$ is a proper complexity function.

Lemma 7.1 $H_f \in \texttt{TIME}(f(n)^3)$ where $n = |M; x|$.
$(H_f \in \texttt{TIME}(f(n) \cdot \lg^2 f(n)))$

Lemma 7.2

$H_f \notin \mathtt{TIME}(f(\lfloor \frac{n}{2} \rfloor))$.

Proof: By contradiction. Suppose M_{H_f} decides H_f in time $f(\lfloor \frac{n}{2} \rfloor)$. Define $D_f(M)$ as

$$\text{if } M_{H_f}(M; M) = \text{“yes” then “no”, else “yes”.}$$

What is $D_f(D_f)$?

If $D_f(D_f) =$ “yes”, then $M_{H_f}(M_{D_f}; M_{D_f}) =$ “no”,
“no” “yes”.

Contradiction!

The Time Hierarchy Theorem

Theorem 7.1

If $f(n) \geq n$ is a proper complexity function, then the class $\mathtt{TIME}(f(n))$ is strictly contained within $\mathtt{TIME}(f(2n+1)^3)$.

Remark

A stronger version suggests that

$$\mathtt{TIME}(f(n)) \subsetneq \mathtt{TIME}(f(n) \lg^2 f(n)).$$

Corollary $\mathcal{P}$ is a proper subset of EXP.

1. $\mathcal{P}$ is a subset of $\mathtt{TIME}(2^n)$.

2. $\mathtt{TIME}(2^n) \subsetneqq \mathtt{TIME}((2^{2n+1})^3)$ (Time Hierarchy Theorem)
   $\mathtt{TIME}((2^{2n+1})^3) \subseteq \mathtt{TIME}(2^{n^2}) \subseteq \text{EXP}$.

The Space Hierarchy Theorem

If $f(n)$ is a proper function, then $\texttt{SPACE}(f(n))$ is a proper subset of $\texttt{SPACE}(f(n) \lg f(n))$.

(Note that the restriction $f(n) \geq n$ is removed from the Time Hierarchy Theorem.)

The Reachability Method

Theorem 7.4 Suppose that $f(n)$ is a proper complexity function.

1. $\texttt{SPACE}(f(n)) \subseteq \texttt{NSPACE}(f(n))$, $\texttt{TIME}(f(n)) \subseteq \texttt{NTIME}(f(n))$. ($\because$ DTM is a special NTM.)
2. $\texttt{NTIME}(f(n)) \subseteq \texttt{SPACE}(f(n))$.
3. $\texttt{NSPACE}(f(n)) \subseteq \texttt{TIME}(k^{\lg n + f(n)})$ for $k > 1$.

Corollary

$$\mathcal{L} \subseteq \mathcal{NL} \subseteq \mathcal{P} \subseteq \mathcal{NP} \subseteq \text{PSPACE}.$$

However, $\mathcal{L} \subsetneq$ PSPACE. Hence at least one of the four inclusions is proper. (Space Hierarchy Theorem)

Theorem 7.5: (Savitch's Theorem)
REACHABILITY $\in \texttt{SPACE}(\lg^2 n)$.

Corollary

1. $\texttt{NSPACE}(f(n)) \subseteq \texttt{SPACE}(f(n)^2)$ for any proper complexity function $f(n) \geq \lg n$.
2. PSPACE $=$ NPSPACE

Immerman-Szelepscényi Theorem

Theorem 7.6 If $f \geq \lg n$ is a proper complexity function, then $\texttt{NSPACE}(f(n)) = co\texttt{NSPACE}(f(n))$.

Corollary $\mathcal{NL} = co\mathcal{NL}$.