Fundamentals of Mathematics

The Midterm Examination

Spring, 2008

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Problem 1 (10 points) Show that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + m(m+1) \geq \frac{m^3}{3} \quad \text{for integers } m \geq 0$$

by the method of mathematical induction.

Problem 2 (10 points) Show that $2^m > m^2$ for $m \geq 5$.

Problem 3 (10 points) Show that

$$\frac{c_1 + \cdots + c_m}{m} \geq (c_1 \cdots c_m)^{\frac{1}{m}} \text{ where } c_i \geq 0 \text{ for } 1 \leq i \leq m.$$

Problem 4 (10 points) Show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{m}} > \sqrt{m}$$

for integers $m \geq 2$.

Problem 5 (10 points) Let $f : A \to B$ for any sets A and B. Let $f^{-1}(T) = \{a |\ f(a) \in T \text{ where } a \in A\}$ for any $T \subseteq B$. Show that

$$f^{-1}(\overline{T}) = \overline{f^{-1}(T)}.$$

Problem 6 (10 points) Let $f : A \to B$ for sets A and B. Let $f(S) = \{f(a) |\ a \in S\}$ for any $S \subseteq A$. Give a counterexample to show that

$$f(\overline{S}) \supseteq \overline{f(S)}$$

is wrong.

Problem 7 (10 points) Let m and n be any natural numbers. Show that

$$\gcd(m, n) = \gcd(n, m \mod n)$$

where gcd is the function that evaluates the greatest common divisor of its arguments.

Problem 8 (10 points) Find all solutions of this equation:

$$\frac{4}{15} = \frac{1}{m} + \frac{1}{n} \qquad \text{where } m \text{ and } n \text{ are natural numbers.}$$

Problem 9 (10 points) For all integers m, show that the following statements are equivalent:

(1) m is even;

(2) m^2 is even;

(3) m^k is even for all integers $k \geq 1$.

Problem 10 (10 points) Derive the negation of the following statement:

$$\forall_{n \geq 0} \exists_{m \geq n} \left[P(m,n) \implies \neg Q(m,n) \right].$$