Fundamentals of Mathematics

Final Examination

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Problem 1 (20 points)

1. Let $p(x)$ be a uni-variable integral polynomial. That is, we can write $p(x)$ as $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ where $a_i's$ are integers and n is its degree. Show that the number of roots (solutions) of $p(x) = 0$ is at most n.

2. Show that the number of all such polynomials is countable.

3. Show that there exists a real number that is not the root of any integral polynomial.

Problem 2 (20 points) Show that the cardinalities of $\mathbb{R}$ and the open interval $(0,1)$ are equal.

Problem 3 (20 points) Let $A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. That is, we remove the open middle third $(\frac{1}{3}, \frac{2}{3})$ from the interval $[0,1]$. Let

$$A_{n+1} = \frac{A_n}{3} \cup (\frac{2}{3} + \frac{A_n}{3}) \quad \text{for } n \geq 1 \ .$$

Intuitively speaking, A_{n+1} is obtained from A_n by removing every open middle third from all intervals of A_n. Let $A_\infty = \lim_{n\to\infty} A_n$.

1. Show that the 'length' of A_∞ is zero. That is, the total length of the removed intervals of A_∞ from $[0,1]$ equals to 1.

2. Show that the cardinality of A_∞ is uncountable.

(Note: You can get more information by looking up the keyword 'Cantor Set' from Wikipedia.)

Problem 4 (20 points) Compare the growth rate of

$$2^n \text{ and } \binom{n}{\frac{n}{3}}$$

as n tends to infinity. (Note: Stirling formula asserts $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$.)