Fundamentals of Mathematics

Homework Set 2

Spring, 2008

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Due date: March 10

Problem 1 Please answer the following questions.

1. What is x^0 for $x \neq 0$?
2. What is 0^x for $x > 0$?
3. What is 0^0?

Problem 2 Let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$. How many functions from $\mathbb{Z}_n$ to $\mathbb{Z}_n$ such that $f(k) \neq k$ for all $k \in \mathbb{Z}_n$? Let this set be F_n.

1. Let $A_i = \{f |\ f : \mathbb{Z}_n \to \mathbb{Z}_n$ with $f(i) = i\}$ for all $i \in \mathbb{Z}_n$. Show that
$$F_n = \bigcap_{i \in \mathbb{Z}_n} \overline{A_i}.$$
2. Let χ_{A_i} be the indicator function of A_i for $i \in \mathbb{Z}_n$. Show that
$$\Pr(f \in \overline{A_i}) = \Pr(\mathbf{1} - \chi_{A_i}) = 1 - \frac{1}{n}.$$
3. Show that
$$\Pr(f \in F_n) = (1 - \frac{1}{n})^n.$$
4. Let $n \to \infty$. Show that $\Pr(f \in F_\infty) = e^{-1}$. That is, with probability e^{-1}, a random function maps from natural numbers to natural numbers with no *fixed point*.

Problem 3 Translate 2008 into the corresponding Roman numeral.

Problem 4 Let $f : A \to B$ for any sets A and B. Let $f(S) = \{f(a) |\ a \in S\}$ for any $S \subseteq A$. Let $f^{-1}(T) = \{a |\ f(a) \in T$ for some $a \in A\}$ for any $T \in B$. Show that

1. $f(\overline{S}) \supseteq \overline{f(S)}$;
2. $f^{-1}(\overline{T}) = \overline{f^{-1}(T)}$;
3. $f^{-1}(f(S)) \supseteq S$;
4. $f(f^{-1}(T)) \subseteq T$.

Give examples for Cases 1, 3, 4 to illustrate when equality may not hold.