Fundamentals of Mathematics
Lecture 8: Set Theory

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Spring, 2008

Cardinality

Let A and B be two sets. We say

1. $\mathrm{Card}(A) = \mathrm{Card}(B)$ iff there is a bijection between A and B.
2. $\mathrm{Card}(A) \leq \mathrm{Card}(B)$ iff there is an injection from A to B.
3. $\mathrm{Card}(A) \geq \mathrm{Card}(B)$ iff there is a surjection from A to B.

Theorem (Schroeder-Bernstein)

If $\mathrm{Card}(A) \leq \mathrm{Card}(B)$ *and* $\mathrm{Card}(A) \geq \mathrm{Card}(B)$, *then* $\mathrm{Card}(A) = \mathrm{Card}(B)$.

Finite Set

- A set is called finite iff there exists integers $n \geq 0$ such that $\mathrm{Card}(A) = \mathrm{Card}(\{1, 2, \ldots, n\})$.
- A set is called infinite iff it is not finite.

Denumerable Set

- A set is called denumerable iff

$$\mathrm{Card}(A) = \mathrm{Card}(\mathbb{N}) \quad .$$

- A set is called countable iff it is either finite of denumerable.

Theorem (Dedekind-Peirce)

A set is infinite if and only if it has a bijection with a proper subset of itself.

Theorem

- *The set $\mathbb{Z}$ is denumerable.* $(n \leftrightarrow 2n+1, \; (-n) \leftrightarrow 2n)$
- *The set $\mathbb{Q}$ is denumerable.*

Theorem (Georg Cantor)

The set $\mathbb{R}$ of all real numbers is not countable.

Proof.

$\square$

Power Set I

2^A denotes the set of all subsets of A.

Theorem

$\mathrm{Card}(A) < \mathrm{Card}(2^A)$ *for any set* A. *($\leq$ but not $=$)*

Proof.

Suppose there is a bijection between A and 2^A: $a \leftrightarrow f(a) \in 2^A$. Define C as

$$\{a |\ a \in A \text{ and } a \notin f(a)\} \ .$$

Then C is a subset of A and thus there exists $\alpha \in A$ such that $f(\alpha) = C$.

- Suppose $\alpha \in C$. By the definition of C, $\alpha \notin C$.
- Suppose $\alpha \notin C$. By the definition of C, $\alpha \in C$.

All of the cases lead to a contradiction. □

Power Set II

Corollary (Bertrand Russel)

There is no set of all sets.

More Properties

1. $\mathrm{Card}(2^{\mathbb{Z}}) = \mathrm{Card}(\mathbb{R})$
2. $\mathrm{Card}(\mathbb{N}^k) = \mathrm{Card}(\mathbb{N})$ for any $k \in \mathbb{N}$
3. Let Σ be a countable set. Let $\Sigma^* = \{x |\ x$ is a finite string over $\Sigma\}$
   Then $\mathrm{Card}(\Sigma^*) = \mathrm{Card}(\mathbb{N})$
4. $\mathrm{Card}(\mathbb{R}^k) = \mathrm{Card}(\mathbb{R})$ for any $k \in \mathbb{N}$

Let $\Sigma = \{0, 1\}$. $L \subseteq \Sigma^*$ is called a language over Σ.

- Σ^* is countable (denumerable).
- The set of all languages over Σ is 2^{Σ^*}, which is uncountable.
- A program is a finite string. The set of all programs is countable.
- There exists a language L such that there is no program that can answer whether $x \in L$ for any $x \in \Sigma^*$.

References

P. R. Halmos, Naive Set Theory, Springer-Verlag, 1974.