Fundamentals of Mathematics
Lecture 6: Propositional Logic

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Propositional Connectives I

1. Negation: $\neg$ (not A)

A	$\neg A$
T	F
F	T

2. Conjunction: $\wedge$ (A and B)

A	B	$A \wedge B$
T	T	T
F	T	F
T	F	F
F	F	F

3. Disjunction: $\vee$ (A or B)

A	B	$A \vee B$
T	T	T
F	T	T
T	F	T
F	F	F

Propositional Connectives II

4. Conditional: $\Rightarrow$ (if A, then B)

A	B	$A \Rightarrow B$
T	T	T
F	T	T
T	F	F
F	F	T

Remark

The above definition for $\Rightarrow$ is only appropriate for mathematics. Consider the following cases.

- *If this piece of iron is placed in water at time t, then the iron will dissolve. (causal laws)*
- *If you were not born, there would be no 921 earthquake in Taiwan. (counter factual)*

Propositional Connectives III

5. Biconditional: $\Leftrightarrow$ (A if and only if B)

A	B	$A \Leftrightarrow B$
T	T	T
F	T	F
T	F	F
F	F	T

Propositional Connectives IV

Definition (Statement form)

1. All statement letters (capital italic letters, e.g., A, B, C) and such letters with numerical subscripts (e.g., A_1, C_5) are statement forms.
2. If $\mathcal{A}$ and $\mathcal{B}$ are statement forms, then so are $(\neg\mathcal{A})$, $(\mathcal{A} \wedge \mathcal{B})$, $(\mathcal{A} \vee \mathcal{B})$, $(\mathcal{A} \Rightarrow \mathcal{B})$, and $(\mathcal{A} \Leftrightarrow \mathcal{B})$.
3. Only those expressions are statement forms that are determined to be so by means of Conditions (1) and (2).

Examples

B, $(\neg C_2)$, $(D_3 \wedge (\neg B))$, $(((\neg B_1) \vee B_2) \Rightarrow A_1 \wedge C_2)$

Truth Tables

Let $\mathcal{A}$ be a statement form. If we are given the truth values of all statement letters of $\mathcal{A}$, the truth value of $\mathcal{A}$ is determined and can be calculated.

1. $(((\neg A) \vee B) \Rightarrow C)$
2. $((A \Leftrightarrow B) \Rightarrow ((\neg A) \wedge B))$

Truth Functions (Boolean Functions)

Definition

A truth function of n arguments is a mapping $\{T, F\}^n \to \{T, F\}$.

Observations

1. *A statement form with n statement letters can be considered as a truth function.*
2. *Conversely, any truth function of n arguments can be expressed in the statement form.*

Proposition

Let $f : \{T, F\}^n \to \{T, F\}$. Then there exists a statement form $A = \mathcal{A}(A_1, \ldots, A_n)$ such that $f(x_1, \ldots, x_n) = \mathcal{A}(A_1, \ldots, A_n)$ whenever $x_1 = A_1, \ldots, x_n = A_n$. (This fact will be proved latter.)

Tautologies I

Definition

A statement form is called a tautology if and only if it is always true, no matter what truth values of its statement letters may be.

Examples

1. $(A \vee (\neg A))$
2. $(A \Leftrightarrow (\neg(\neg A)))$

- If $(\mathcal{A} \Rightarrow \mathcal{B})$ is a tautology, we say $\mathcal{A}$ implies $\mathcal{B}$, or $\mathcal{B}$ is a logical consequence of $\mathcal{A}$.
- If $(\mathcal{A} \Leftrightarrow \mathcal{B})$ is a tautology, we say $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent.

Tautologies II

Examples

- $(A \Rightarrow A \vee B)$
- $(A \Leftrightarrow (\neg(\neg A)))$

Example

Determine whether $((A \Leftrightarrow ((\neg B) \vee C)) \Rightarrow ((\neg A) \Rightarrow B))$ is a tautology. (positive)

Contradictions I

Definition

A statement form is called a contradiction iff it is always false.

Proposition

$\mathcal{A}$ is a tautology if and only if $(\neg\mathcal{A})$ is a contradiction.

Proposition

If $\mathcal{A}$ and $(\mathcal{A} \Rightarrow \mathcal{B})$ are tautologies, then so is $\mathcal{B}$.

Proposition

If $\mathcal{A}(A_1, \ldots, A_n)$ is a tautology, then $\mathcal{A}(A_1 \leftarrow \mathcal{B}_1, \ldots, A_1 \leftarrow \mathcal{B}_n)$ is a tautology. That is, substitution in a tautology yields a tautology.

Contradictions II

Example

Let $\mathcal{A}(A_1, A_2)$ be $((A_1 \wedge A_2) \Rightarrow A_1)$. Set $\mathcal{B}_1$ as $(B \vee C)$ and $\mathcal{B}_2$ as $(C \wedge D)$.

Proposition

Let $\mathcal{B}_1$ be $\mathcal{A}_1(\mathcal{A} \leftarrow \mathcal{B})$. Then $((\mathcal{A} \Leftrightarrow \mathcal{B}) \Rightarrow (\mathcal{A}_1 \Leftrightarrow \mathcal{B}_1))$.

Example

Let $\mathcal{A}_1$ be $(C \vee D)$, $\mathcal{A}$ be C, and $\mathcal{B}$ be $(\neg(\neg C))$.

Parentheses I

Remove unnecessary parentheses by taking the following convention:

1. Omit the outer pair of parentheses of a statement form;
2. Connectives are ordered as follows: $\neg > \wedge > \vee > \Rightarrow > \Leftrightarrow$; (the precedence)
3. $\wedge$ and $\vee$ are left-to-right association; $\Rightarrow$ is right-to-left association; $\Leftrightarrow$ is as an equivalence relation.

Example

$$A \Leftrightarrow \neg B \vee C \Rightarrow A$$
$$A \Leftrightarrow (\neg B) \vee C \Rightarrow A$$
$$A \Leftrightarrow ((\neg B) \vee C) \Rightarrow A$$
$$A \Leftrightarrow (((\neg B) \vee C) \Rightarrow A)$$
$$(A \Leftrightarrow (((\neg B) \vee C) \Rightarrow A))$$

Parentheses II

Example

- $A \wedge B \wedge C$ as $((A \wedge B) \wedge C)$.
- $A \Rightarrow B \Rightarrow C$ as $A \Rightarrow (B \Rightarrow C)$.
- $A \Leftrightarrow B \Leftrightarrow C$ as $(A \Leftrightarrow B) \wedge (B \Leftrightarrow C)$.

Example

$(A \Rightarrow B) \Rightarrow C$ is different from $A \Rightarrow (B \Rightarrow C)$. (Set A, B, C all false.)

Remark

$\wedge$ *and* $\vee$ *are associative and commutative.*

Adequate Set of Connectives I

A set of connectives is called adequate iff every truth function (of finite arguments) has a corresponding statement form under this set of connectives.

Proposition

Every truth function is generated by a statement form involving the connectives $\neg$, $\wedge$, *and* $\vee$.

Proof.

Let f be a truth function of n arguments. Let $f|_{x_1=T}$ be the function $f(T, x_2, \ldots, x_n)$ that sets $x_1 = T$ in f. Similarly, $f|_{x_1=F}$ be the restriction setting $x_1 = F$. Clearly $f|_{x_1=T}$ and $f|_{x_1=F}$ are truth functions of $n-1$ arguments. Then by induction, there are statement forms $\mathcal{A}_1$ and $\mathcal{A}_2$ that represent $f|_{x_1=T}$ and $f|_{x_1=F}$, respectively. It can be verify that $(\mathcal{A}_1 \wedge x_1) \vee (\mathcal{A}_2 \wedge \neg x_1)$ is a statement form that represents f. $\square$

Adequate Set of Connectives II

Corollary

Every truth function corresponds to a statement form containing connectives only $\wedge$ *and* $\neg$, *or only* $\vee$ *and* $\neg$, *or only* $\Rightarrow$ *and* $\neg$.

Adequate Set of Connectives III

Definition

$\downarrow$ (joint denial, NOR)

A	B	$A \downarrow B$
T	T	F
F	T	F
T	F	F
F	F	T

$\mid$ (alternative denial, NAND)

A	B	$A \mid B$
T	T	F
F	T	T
T	F	T
F	F	T

Adequate Set of Connectives IV

Proposition

The only binary connectives that along are adequate for the construction of all truth functions are $\downarrow$ and $|$. (Propositional constants T and F are not allowed in use.)

Adequate Set of Connectives V

Proof.

Let $\circ$ be a binary connective that is adequate. Observe the following properties:

1. $T \circ T$ must be F: Otherwise any statement form constructed by $\circ$ along will be true when all of its statement letters are true.
2. $F \circ F$ must be T: Otherwise any statement form constructed by $\circ$ along will be false when all of its statement letters are false.
3. $F \circ T$ and $T \circ F$ must be the same. Otherwise flipping all arguments causes the change of the output value, which is not always true in all truth functions.

Hence the only candidates are $\downarrow$ (NOR) and $|$ (NAND).
On the other hand, $\neg A$ is equivalent to $A \downarrow A$ and to $A \mid A$;
$(A \downarrow B) \downarrow (A \downarrow B)$ is equivalent to $A \vee B$ and $(A \mid B) \mid (A \mid B)$ is equivalent to $A \wedge B$. Since both $\{\neg, \vee\}$ and $\{\neg, \wedge\}$ are adequate, it follows that $\downarrow$ or $|$ is adequate. $\square$

Formal Theories

A formal theory $\mathcal{T}$ has four parts $(\mathfrak{S}, \mathfrak{F}, \mathfrak{A}, \mathfrak{R})$ where

1. $\mathfrak{S}$: a countable set of symbols
   $x \in \mathfrak{S}^*$, the set of all finite strings over $\mathfrak{S}$, is called an expression.
2. $\mathfrak{F}$: the set of well-formed formulas
   $\mathfrak{F} \subseteq \mathfrak{S}^*$ and whether $x \in \mathfrak{F}$ can be effectively verified.
3. $\mathfrak{A}$: the set of axioms; $\mathfrak{A} \subseteq \mathfrak{F}$ and its membership problem can be effectively verified.
4. $\mathfrak{R}$: rules of inference
   $\mathfrak{R} = \{R_1, R_2, \ldots, R_n\}$ where each R_i is a $k_i + 1$-ary relation over $\mathfrak{F}$ written as

   $$\frac{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{k_i}}{\mathcal{A}_{k_i+1}} \quad .$$

   R_i can be effectively verified when given $\mathcal{A}_1, \ldots, \mathcal{A}_{k_i+1}$. The lower term $\mathcal{A}_{k_i+1}$ is called a direct consequence.

Definition (Proof)

A proof in $\mathcal{T}$ is a sequence

$$\mathcal{A}_1, \ldots, \mathcal{A}_n$$

of wfs such that, for each i, either $\mathcal{A}_i \in \mathfrak{A}$ or is a direct consequence of the preceding wfs by using one of the rules of inference.

Definition (Theorem)

A theorem of $\mathcal{T}$ is a wf $\mathcal{A}$ of $\mathcal{T}$ such that there is a proof where the last wf is $\mathcal{A}$.

Definition (Decidability)

A theory $\mathcal{T}$ is called decidable if there is an algorithm for determining, given any wf $\mathcal{A}$, whether there is a proof of $\mathcal{A}$.
Otherwise, if no such an algorithm does exist, it is called undecidable

Definition (Consequence)

A wf $\mathcal{A}$ is a consequence in $\mathcal{T}$ of a set Γ of wfs iff there is a sequence

$$\mathcal{A}_1, \ldots, \mathcal{A}_n \quad \text{of wfs}$$

such that

1. $\mathcal{A}_n = \mathcal{A}$;
2. for each i, either $\mathcal{A}_i$ is an axiom, or $\mathcal{A}_i \in \Gamma$, or $\mathcal{A}_i$ is a direct consequence of the preceding wfs.

It is written as $\Gamma \vdash \mathcal{A}$.

Remarks

1. *When Γ is a finite set $\{\mathcal{B}_1, \ldots, \mathcal{B}_k\}$, we write $\mathcal{B}_1, \ldots, \mathcal{B}_k \vdash \mathcal{A}$ instead of $\{\mathcal{B}_1, \ldots, \mathcal{B}_k\} \vdash \mathcal{A}$.*
2. *$\vdash \mathcal{A}$ means $\mathcal{A}$ is a theorem of $\mathcal{T}$.*

Observations

1. *If* $\Gamma \subseteq \Delta$ *and* $\Gamma \vdash \mathcal{A}$*, then* $\Delta \vdash \mathcal{A}$*.*
2. $\Gamma \vdash \mathcal{A}$ *if and only if there is a finite subset* $\Delta \subseteq \Delta$ *such that* $\Delta \vdash \mathcal{A}$*.*
3. *If* $\Delta \vdash \mathcal{A}$ *and for each* $\mathcal{B} \in \Delta$ *we have* $\Gamma \vdash \mathcal{B}$*, then* $\Gamma \vdash \mathcal{A}$*.*

Axiomatic Theory for the Propositional Calculus

The formal theory $L = (\mathfrak{S}, \mathfrak{F}, \mathfrak{A}, \mathfrak{R})$ is as follows.

1. $\mathfrak{S} = \{\neg, \Rightarrow, (,)\} \cup \{A_i |\ i \in \mathbb{N}\}$.
   $\neg$ and $\Rightarrow$ are called primitive connectives; A_i are called statement letters.
2. The set of wfs $\mathfrak{F}$ is defined recursively as
   1. all statement letters are wfs;
   2. if $\mathcal{A}$ and $\mathcal{B}$ are wfs, so are $(\neg\mathcal{A})$ and $(\mathcal{A} \Rightarrow \mathcal{B})$.
3. Let $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ be any wfs of L. $\mathfrak{A}$ contains
   (A1) $(\mathcal{A} \Rightarrow (\mathcal{B} \Rightarrow \mathcal{A}))$.
   (A2) $((\mathcal{A} \Rightarrow (\mathcal{B} \Rightarrow \mathcal{C})) \Rightarrow ((\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow (\mathcal{A} \Rightarrow \mathcal{C})))$.
   (A3) $(((\neg\mathcal{B}) \Rightarrow (\neg\mathcal{A})) \Rightarrow (((\neg\mathcal{B}) \Rightarrow \mathcal{A}) \Rightarrow \mathcal{B}))$.
4. The only inference rule is modus ponens

$$\frac{\mathcal{A}, (\mathcal{A} \Rightarrow \mathcal{B})}{\mathcal{B}} \quad \text{(MP)} \ .$$

Remarks

(D1) $(\mathcal{A} \wedge \mathcal{B})$ *for* $\neg(\mathcal{A} \Rightarrow \neg\mathcal{B})$

(D2) $(\mathcal{A} \vee \mathcal{B})$ *for* $(\neg\mathcal{A}) \Rightarrow \mathcal{B}$

(D3) $(\mathcal{A} \Leftrightarrow \mathcal{B})$ *for* $(\mathcal{A} \Rightarrow \mathcal{B}) \wedge (\mathcal{B} \Rightarrow \mathcal{A})$

Lemma

$\vdash_L \mathcal{A} \Rightarrow \mathcal{A}$ *for all wfs* $\mathcal{A}$.

Proof.

1. $(\mathcal{A} \Rightarrow ((\mathcal{A} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{A})) \Rightarrow ((\mathcal{A} \Rightarrow (\mathcal{A} \Rightarrow \mathcal{A})) \Rightarrow (\mathcal{A} \Rightarrow \mathcal{A}))$ (Axiom (A2))
2. $\mathcal{A} \Rightarrow ((\mathcal{A} \Rightarrow \mathcal{A}) \Rightarrow A)$ (Axiom (A1))
3. $(\mathcal{A} \Rightarrow (\mathcal{A} \Rightarrow \mathcal{A})) \Rightarrow (\mathcal{A} \Rightarrow \mathcal{A})$ (from 1 and 2 by MP)
4. $\mathcal{A} \Rightarrow (\mathcal{A} \Rightarrow A)$ (Axiom (A1))
5. $\mathcal{A} \Rightarrow \mathcal{A}$ (from 3 and 4 by MP)

$\square$

Proposition (Deduction Theorem)

If $\Gamma, \mathcal{A} \vdash \mathcal{B}$, *then* $\Gamma \vdash \mathcal{A} \Rightarrow \mathcal{B}$. *In particular, if* $\mathcal{A} \vdash \mathcal{B}$, *then* $\vdash \mathcal{A} \Rightarrow \mathcal{B}$.

Let $\mathcal{B}_1, \ldots, \mathcal{B}_n$ be a proof of $\mathcal{B}$ from $\Gamma \cup \{\mathcal{A}\}$ where $\mathcal{B}_n = \mathcal{B}$. We will show, by induction, $\Gamma \vdash \mathcal{A} \Rightarrow \mathcal{B}_i$ for $1 \leq i \leq n$.

1. (Basis) $\mathcal{B}_1$ must be either in Γ or an axiom of L or $\mathcal{A}$ itself.
    1. $\mathcal{B}_1 \in \Gamma$ or $\mathcal{B}_1$ is an axiom: $\mathcal{B}_1$, $\mathcal{B}_1 \Rightarrow \mathcal{A} \Rightarrow \mathcal{B}_1$, $\mathcal{A} \Rightarrow \mathcal{B}_1$ is a proof.
    2. $\mathcal{B}_1 = \mathcal{A}$: $\mathcal{A} \Rightarrow \mathcal{A}$ is proved on Page 25.
2. (Induction) $\mathcal{B}_i$ is either in Γ or an axiom of L or $\mathcal{A}$ itself or by MP for $1 < i \leq n$.
    1. The first three cases are the same as for $i = 1$.
    2. $\mathcal{B}_i$ is the direct consequence of $\mathcal{B}_j$ and $\mathcal{B}_m = \mathcal{B}_j \Rightarrow \mathcal{B}_i$ by MP where $j < i$ and $m < i$. By induction, we have $\Gamma \vdash \mathcal{A} \Rightarrow \mathcal{B}_j$ and $\Gamma \vdash \mathcal{A} \Rightarrow (\mathcal{B}_j \Rightarrow \mathcal{B}_i)$. By Axiom (A2), we have $\vdash (\mathcal{A} \Rightarrow (\mathcal{B}_j \Rightarrow \mathcal{B}_i)) \Rightarrow ((\mathcal{A} \Rightarrow \mathcal{B}_j) \Rightarrow (\mathcal{A} \Rightarrow \mathcal{B}_i))$. By twice applications of MP, we get a proof of $\mathcal{A} \Rightarrow \mathcal{B}_i$.

Corollaries

- $\mathcal{A} \Rightarrow \mathcal{B}, \mathcal{B} \Rightarrow \mathcal{C} \vdash \mathcal{A} \Rightarrow \mathcal{C}$
- $\mathcal{A} \Rightarrow (\mathcal{B} \Rightarrow \mathcal{C}), \mathcal{B} \vdash \mathcal{A} \Rightarrow \mathcal{C}$

Lemmas I

For any wfs $\mathcal{A}$ amd $\mathcal{B}$, the followings are theorems of L.

(a) $\neg\neg\mathcal{B} \Rightarrow \mathcal{B}$

Proof.

1. $(\neg\mathcal{B} \Rightarrow \neg\neg\mathcal{B}) \Rightarrow ((\neg\mathcal{B} \Rightarrow \neg\mathcal{B}) \Rightarrow \mathcal{B})$ (Axiom schema (A3))
2. $\neg\mathcal{B} \Rightarrow \neg B$ (the lemma on Page 25)
3. $(\neg\mathcal{B} \Rightarrow \neg\neg\mathcal{B}) \Rightarrow \mathcal{B}$ (1,2, and the corollary on Page 27)
4. $\neg\neg\mathcal{B} \Rightarrow (\neg B \Rightarrow \neg\neg\mathcal{B})$ (Axiom (A1))
5. $\neg\neg\mathcal{B} \Rightarrow \mathcal{B}$ (3, 4, the corollary on Page 27)

$\square$

Lemmas II

(b) $\mathcal{B} \Rightarrow \neg\neg\mathcal{B}$

Proof.

1. $(\neg\neg\neg\mathcal{B} \Rightarrow \neg\mathcal{B}) \Rightarrow ((\neg\neg\neg\mathcal{B} \Rightarrow \mathcal{B}) \Rightarrow \neg\neg\mathcal{B})$ (Axiom (A3))
2. $\neg\neg\neg\mathcal{B} \Rightarrow \neg\mathcal{B}$ (Part (a))
3. $(\neg\neg\neg\mathcal{B} \Rightarrow \mathcal{B}) \Rightarrow \neg\neg\mathcal{B}$ (1, 2, MP)
4. $\mathcal{B} \Rightarrow (\neg\neg\neg\mathcal{B} \Rightarrow \mathcal{B})$ (Axiom (A1))
5. $\mathcal{B} \Rightarrow \neg\neg\mathcal{B}$ (3, 4, the corollary on Page 27)

$\square$

Lemmas III

(c) $\neg\mathcal{A} \Rightarrow (\mathcal{A} \Rightarrow \mathcal{B})$

1. $\neg\mathcal{A}$ (Hyp)
2. $\mathcal{A}$ (Hyp)
3. $\mathcal{A} \Rightarrow (\neg\mathcal{B} \Rightarrow \mathcal{A})$ (Axiom (A1))
4. $\neg\mathcal{A} \Rightarrow (\neg\mathcal{B} \Rightarrow \neg\mathcal{A})$ (Axiom (A1))
5. $\neg\mathcal{B} \Rightarrow \mathcal{A}$ (2, 3, MP)
6. $\neg\mathcal{B} \Rightarrow \neg\mathcal{A}$ (1, 4, MP)
7. $(\neg\mathcal{B} \Rightarrow \neg\mathcal{A}) \Rightarrow ((\neg\mathcal{B} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{B})$ (Axiom (A3))
8. $(\neg\mathcal{B} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{B}$ (6, 7, MP)
9. $\mathcal{B}$ (MP)
10. $\neg\mathcal{A}, \mathcal{A} \vdash \mathcal{B}$ (1–9)
11. $\neg\mathcal{A} \vdash \mathcal{A} \Rightarrow \mathcal{B}$ (10, Deduction Theorem)
12. $\vdash \neg\mathcal{A} \Rightarrow (\mathcal{A} \Rightarrow \mathcal{B})$ (11, Deduction Theorem)

Lemmas IV

(d) $(\neg\mathcal{B} \Rightarrow \neg\mathcal{A}) \Rightarrow (\mathcal{A} \Rightarrow \mathcal{B})$

Proof.

1. $\neg\mathcal{B} \Rightarrow \neg\mathcal{A}$ (Hyp)
2. $(\neg\mathcal{B} \Rightarrow \neg\mathcal{A}) \Rightarrow ((\neg\mathcal{B} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{B})$ (Axiom (A3))
3. $\mathcal{A} \Rightarrow (\neg\mathcal{B} \Rightarrow \mathcal{A})$ (Axiom (A1))
4. $(\neg\mathcal{B} \Rightarrow \mathcal{A}) \Rightarrow \mathcal{B}$ (1, 2, MP)
5. $\mathcal{A} \Rightarrow \mathcal{B}$ (3,4, the corollary on Page 27)
6. $\neg\mathcal{B} \Rightarrow \neg\mathcal{A} \vdash \mathcal{A} \Rightarrow \mathcal{B}$ (1–5)
7. $\vdash (\neg\mathcal{B} \Rightarrow \neg\mathcal{A}) \Rightarrow (\mathcal{A} \Rightarrow \mathcal{B})$ (6, Deduction Theorem)

$\square$

Lemmas V

(e) $\vdash (\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow (\neg\mathcal{B} \Rightarrow \neg\mathcal{A})$

1. $\mathcal{A} \Rightarrow \mathcal{B}$ (Hyp)
2. $\neg\neg\mathcal{A} \Rightarrow \mathcal{A}$ (Part(a))
3. $\neg\neg\mathcal{A} \Rightarrow \mathcal{B}$ (1, 2, the corollary on Page 27)
4. $\mathcal{B} \Rightarrow \neg\neg\mathcal{B}$ (Part(b))
5. $\neg\neg\mathcal{A} \Rightarrow \neg\neg\mathcal{B}$ (3, 4, the corollary on Page 27)
6. $(\neg\neg\mathcal{A} \Rightarrow \neg\neg\mathcal{B}) \Rightarrow (\neg\mathcal{B} \Rightarrow \neg\mathcal{A})$ (Part(d))
7. $(\neg\mathcal{B} \Rightarrow \neg\mathcal{A})$ (5, 6, MP)
8. $\mathcal{A} \Rightarrow \mathcal{B} \vdash (\neg\mathcal{B} \Rightarrow \neg\mathcal{A})$ (1–7)
9. $\vdash (\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow (\neg\mathcal{B} \Rightarrow \neg\mathcal{A})$ (8, Deduction Theorem)

Lemmas VI

(f) $\vdash \mathcal{A} \Rightarrow (\neg\mathcal{B} \Rightarrow \neg(\mathcal{A} \Rightarrow \mathcal{B}))$

Proof.

1. $\mathcal{A}, \mathcal{A} \Rightarrow \mathcal{B} \vdash \mathcal{B}$ (MP)
2. $\vdash \mathcal{A} \Rightarrow ((\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{B})$ (Deduction Theorem)
3. $\vdash ((\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{B}) \Rightarrow (\neg\mathcal{B} \Rightarrow \neg(\mathcal{A} \Rightarrow \mathcal{B}))$ (Part (e))
4. $\vdash \mathcal{A} \Rightarrow (\neg\mathcal{B} \Rightarrow \neg(\mathcal{A} \Rightarrow \mathcal{B}))$ (the corollary on Page 27)

$\square$

Lemmas VII

(g) $\vdash (\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow ((\neg\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{B})$

1. $\mathcal{A} \Rightarrow \mathcal{B}$ (Hyp)
2. $\neg\mathcal{A} \Rightarrow \mathcal{B}$ (Hyp)
3. $(\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow (\neg\mathcal{B} \Rightarrow \neg\mathcal{A})$ (Part(e))
4. $\neg\mathcal{B} \Rightarrow \neg\mathcal{A}$ (1, 3, MP)
5. $(\neg\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow (\neg\mathcal{B} \Rightarrow \neg\neg\mathcal{A})$ (Part (e))
6. $\neg\mathcal{B} \Rightarrow \neg\neg\mathcal{A}$ (2, 5, MP)
7. $(\neg\mathcal{B} \Rightarrow \neg\neg\mathcal{A}) \Rightarrow ((\neg\mathcal{B} \Rightarrow \neg\mathcal{A}) \Rightarrow \mathcal{B})$ (Axiom (A3))
8. $(\neg\mathcal{B} \Rightarrow \neg\mathcal{A}) \Rightarrow \mathcal{B}$ (6, 7, MP)
9. $\mathcal{B}$ (4, 8, MP)
10. $\vdash (\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow ((\neg\mathcal{A} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{B})$ (Deduction Theorem)

Soundness Theorem

Proposition (Soundness)

Every theorem of L is a tautology.

Proof.

1. All axioms of L are tautologies.
2. The modus ponens of two tautologies is again a tautology.

□

Lemma

Let $\mathcal{A}$ be a wf, and let $B_1, \ldots, B_k$ be the statement letters in $\mathcal{A}$. For any assignment of truth values to $B_1, \ldots, B_k$, define

- $B_i' = B_i$ *if* B_i *takes the value* T*; otherwise* $B_i' = \neg B_i$ *if* B_i *takes the value* F*;*
- $\mathcal{A}' = \mathcal{A}$ *if* $\mathcal{A}$ *takes the value* T*; otherwise* $\mathcal{A}' = \neg\mathcal{A}$ *if* $\mathcal{A}$ *takes the value* F*.*

Then

$$\mathcal{B}_1', \ldots, \mathcal{B}_k' \vdash \mathcal{A}' \ .$$

Example

Let $\mathcal{A}$ be $\neg(\neg A_2 \Rightarrow A_5)$. We have

$$A_2, \neg A_5 \vdash \neg\neg(\neg A_2 \Rightarrow A_5) \ .$$

We show this by induction on the structure of $\mathcal{A}$.

(Basis) $\mathcal{A}$ is a statement letter B_1. It reduces to show that $B_1 \vdash B_1$ and $\neg B_1 \vdash \neg B_1$, which are proved on Page 25.

(Induction)

1. $\mathcal{A}$ is $\neg\mathcal{B}$:
   1. $\mathcal{A}$ takes the value F and $\mathcal{B}$ takes the value T. By induction, we have $B_1', \ldots, B_k' \vdash \mathcal{B}$. By the lemma on Page 29, we have $\vdash \mathcal{B} \Rightarrow \neg\neg\mathcal{B}$. Thus, $B_1', \ldots, B_k' \vdash \neg\mathcal{A}$.
   2. $\mathcal{A}$ takes the value T and $\mathcal{B}$ takes the value F. By induction, we have $B_1', \ldots, B_k' \vdash \neg\mathcal{B}$, which is equal to $B_1', \ldots, B_k' \vdash \mathcal{A}$.
2. $\mathcal{A}$ is $\mathcal{B} \Rightarrow \mathcal{C}$:
   1. $\mathcal{B}$ takes F. We have $B_1', \ldots, B_k' \vdash \neg\mathcal{B}$. By the lemma on Page 30 ($\vdash \neg\mathcal{B} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C}$), we have $B_1', \ldots, B_k' \vdash \mathcal{B} \Rightarrow \mathcal{C}$, and $\mathcal{B} \Rightarrow \mathcal{C}$ is $\mathcal{A}'$.
   2. $\mathcal{C}$ takes T. We have $B_1', \ldots, B_k' \vdash \mathcal{C}$. Then, by axiom (A1) ($\mathcal{C} \Rightarrow \mathcal{B} \Rightarrow \mathcal{C}$), $B_1', \ldots, B_k' \vdash \mathcal{B} \Rightarrow \mathcal{C}$, and $\mathcal{B} \Rightarrow \mathcal{C}$ is $\mathcal{A}'$.
   3. $\mathcal{B}$ takes T and $\mathcal{C}$ takes F. We have $B_1', \ldots, B_k' \vdash \mathcal{B}$ and $B_1', \ldots, B_k' \vdash \neg\mathcal{C}$. Then by the lemma on Page 33 ($\mathcal{B} \Rightarrow (\neg\mathcal{C} \Rightarrow \neg(\mathcal{B} \Rightarrow \mathcal{C}))$), we have $B_1', \ldots, B_k' \vdash \neg(\mathcal{B} \Rightarrow \mathcal{C})$, and $\neg(\mathcal{B} \Rightarrow \mathcal{C})$ is $\mathcal{A}'$.

Completeness Theorem

Proposition (Completeness Theorem)

If a wf $\mathcal{A}$ of L is a tautology, then it is a theorem of L. That is, $\mathcal{A}$ is a tautology implies $\vdash_L \mathcal{A}$.

Let $B_1, \ldots, B_k$ be statement letters in $\mathcal{A}$. Then

$$B_1', \ldots, B_{k-1}', B_k \vdash \mathcal{A} \ ;$$

$$B_1', \ldots, B_{k-1}', \neg B_k \vdash \mathcal{A} \ .$$

Hence

$$B_1', \ldots, B_{k-1}' \vdash \mathcal{B}_k \Rightarrow \mathcal{A} \ ;$$

$$B_1', \ldots, B_{k-1}' \vdash \neg B_k \Rightarrow \mathcal{A} \ .$$

Then by the lemma on Page 34, we have $B_1', \ldots, B_{k-1}' \vdash \mathcal{A}$. This process can be continued, and finally we will get $\vdash \mathcal{A}$.

References

C. L. Chang and R. C. T. Lee, Symbolic Logic and Mechanical Theorem Proving, Academic Press, 1973.

E. Mendelson, Introduction to Mathematical Logic, 3rd edition, Wadsworth, 1987.