Propositional Connectives I

1. **Negation**: \( \neg (\text{not } A) \)

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<tr>
<th></th>
<th>( A )</th>
<th>( \neg A )</th>
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<tbody>
<tr>
<td>T</td>
<td>T</td>
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<td>F</td>
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2. **Conjunction**: \( \land (A \text{ and } B) \)

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<th>( A )</th>
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<th>( A \land B )</th>
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3. **Disjunction**: \( \lor (A \text{ or } B) \)

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<th>( A )</th>
<th>( B )</th>
<th>( A \lor B )</th>
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Propositional Connectives II

**Conditional:** \( \Rightarrow \) (if \( A \), then \( B \))

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**Remark**

The above definition for \( \Rightarrow \) is only appropriate for mathematics. Consider the following cases.

- *If this piece of iron is placed in water at time \( t \), then the iron will dissolve.* (causal laws)
- *If you were not born, there would be no 921 earthquake in Taiwan.* (counter factual)
Propositional Connectives III

5 Biconditional: \( \iff (A \text{ if and only if } B) \)

\[
\begin{array}{ccc}
A & B & A \iff B \\
T & T & T \\
F & T & F \\
T & F & F \\
F & F & T \\
\end{array}
\]
Connectives

Propositional Connectives IV

Definition (Statement form)

1. All statement letters (capital italic letters, e.g., $A$, $B$, $C$) and such letters with numerical subscripts (e.g., $A_1$, $C_5$) are statement forms.

2. If $A$ and $B$ are statement forms, then so are $(\neg A)$, $(A \land B)$, $(A \lor B)$, $(A \rightarrow B)$, and $(A \leftrightarrow B)$.

3. Only those expressions are statement forms that are determined to be so by means of Conditions (1) and (2).

Examples

$B$, $(\neg C_2)$, $(D_3 \land (\neg B))$, $(((\neg B_1) \lor B_2) \Rightarrow A_1 \land C_2)$
Let $\mathcal{A}$ be a statement form. If we are given the truth values of all statement letters of $\mathcal{A}$, the truth value of $\mathcal{A}$ is determined and can be calculated.

1. $(((\neg A) \vee B) \Rightarrow C)$
2. $((A \leftrightarrow B) \Rightarrow ((\neg A) \wedge B))$
Truth Functions (Boolean Functions)

Definition

A truth function of $n$ arguments is a mapping $\{T, F\}^n \rightarrow \{T, F\}$.

Observations

1. A statement form with $n$ statement letters can be considered as a truth function.
2. Conversely, any truth function of $n$ arguments can be expressed in the statement form.

Proposition

Let $f : \{T, F\}^n \rightarrow \{T, F\}$. Then there exists a statement form $A = A(A_1, \ldots, A_n)$ such that $f(x_1, \ldots, x_n) = A(A_1, \ldots, A_n)$ whenever $x_1 = A_1, \ldots, x_n = A_n$. (This fact will be proved latter.)
Tautologies I

Definition

A statement form is called a tautology if and only if it is always true, no matter what truth values of its statement letters may be.

Examples

1. \((A \lor (\neg A))\)
2. \((A \iff (\neg (\neg A)))\)

- If \((A \Rightarrow B)\) is a tautology, we say \(A\) implies \(B\), or \(B\) is a logical consequence of \(A\).
- If \((A \Leftrightarrow B)\) is a tautology, we say \(A\) and \(B\) are logically equivalent.
Tautologies II

Examples

- \((A \Rightarrow A \lor B)\)
- \((A \Leftrightarrow (\neg (\neg A)))\)

Example

Determine whether \(((A \Leftrightarrow ((\neg B) \lor C)) \Rightarrow ((\neg A) \Rightarrow B))\) is a tautology.
(positive)
Definition
A statement form is called a contradiction iff it is always false.

Proposition
\( \mathcal{A} \) is a tautology if and only if \((\neg \mathcal{A})\) is a contradiction.

Proposition
If \( \mathcal{A} \) and \((\mathcal{A} \Rightarrow \mathcal{B})\) are tautologies, then so is \( \mathcal{B} \).

Proposition
If \( \mathcal{A}(A_1, \ldots, A_n) \) is a tautology, then \( \mathcal{A}(A_1 \leftarrow B_1, \ldots, A_1 \leftarrow B_n) \) is a tautology. That is, substitution in a tautology yields a tautology.
Contradictions II

Example
Let $A(A_1, A_2)$ be $((A_1 \land A_2) \Rightarrow A_1)$. Set $B_1$ as $(B \lor C)$ and $B_2$ as $(C \land D)$.

Proposition
Let $B_1$ be $A_1(A \leftarrow B)$. Then $((A \leftrightarrow B) \Rightarrow (A_1 \leftrightarrow B_1))$.

Example
Let $A_1$ be $(C \lor D)$, $A$ be $C$, and $B$ be $(\neg(\neg C))$. 
Parentheses I

Remove unnecessary parentheses by taking the following convention:

1. Omit the outer pair of parentheses of a statement form;
2. Connectives are ordered as follows: \( \neg \) > \( \land \) > \( \lor \) > \( \Rightarrow \) > \( \Leftrightarrow \); (the precedence)
3. \( \land \) and \( \lor \) are left-to-right association; \( \Rightarrow \) is right-to-left association; \( \Leftrightarrow \) is as an equivalence relation.

Example

\[
A \Leftrightarrow \neg B \lor C \Rightarrow A
\]
\[
A \Leftrightarrow (\neg B) \lor C \Rightarrow A
\]
\[
A \Leftrightarrow ((\neg B) \lor C') \Rightarrow A
\]
\[
A \Leftrightarrow (((\neg B) \lor C') \Rightarrow A)
\]
\[
(A \Leftrightarrow (((\neg B) \lor C') \Rightarrow A))
\]
Parentheses II

Example
1. $A \land B \land C$ as $((A \land B) \land C)$.
2. $A \Rightarrow B \Rightarrow C$ as $A \Rightarrow (B \Rightarrow C)$.
3. $A \Leftrightarrow B \Leftrightarrow C$ as $(A \Leftrightarrow B) \land (B \Leftrightarrow C)$.

Example
$(A \Rightarrow B) \Rightarrow C$ is different from $A \Rightarrow (B \Rightarrow C)$. (Set $A, B, C$ all false.)

Remark
$\land$ and $\lor$ are associative and commutative.
Adequate Sets

Adequate Set of Connectives I

A set of connectives is called adequate iff every truth function (of finite arguments) has a corresponding statement form under this set of connectives.

Proposition

Every truth function is generated by a statement form involving the connectives \( \neg \), \( \land \), and \( \lor \).

Proof.

Let \( f \) be a truth function of \( n \) arguments. Let \( f|_{x_1=T} \) be the function \( f(T, x_2, \ldots, x_n) \) that sets \( x_1 = T \) in \( f \). Similarly, \( f|_{x_1=F} \) be the restriction setting \( x_1 = F \). Clearly \( f|_{x_1=T} \) and \( f|_{x_1=F} \) are truth functions of \( n - 1 \) arguments. Then by induction, there are statement forms \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) that represent \( f|_{x_1=T} \) and \( f|_{x_1=F} \), respectively. It can be verify that \((\mathcal{A}_1 \land x_1) \lor (\mathcal{A}_2 \land \neg x_1)\) is a statement form that represents \( f \). \( \square \)
Corollary

*Every truth function corresponds to a statement form containing connectives only $\land$ and $\neg$, or only $\lor$ and $\neg$, or only $\Rightarrow$ and $\neg$.***
**Definition**

\(\downarrow\) (joint denial, NOR)

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<tr>
<th>A</th>
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<th>A (\downarrow) B</th>
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<table>
<thead>
<tr>
<th>(alternative denial, NAND)</th>
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<tr>
<td>(A \mid B)</td>
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<tr>
<td>----------------------------</td>
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<tr>
<td>A</td>
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<td>F</td>
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Proposition

The only binary connectives that along are adequate for the construction of all truth functions are $\downarrow$ and $|$. (Propositional constants $T$ and $F$ are not allowed in use.)
Proof.

Let $\circ$ be a binary connective that is adequate. Observe the following properties:

1. $T \circ T$ must be $F$: Otherwise any statement form constructed by $\circ$ along will be true when all of its statement letters are true.
2. $F \circ F$ must be $T$: Otherwise any statement form constructed by $\circ$ along will be false when all of its statement letters are false.
3. $F \circ T$ and $T \circ F$ must be the same. Otherwise flipping all arguments causes the change of the output value, which is not always true in all truth functions.

Hence the only candidates are $\downarrow$ (NOR) and $|$ (NAND).

On the other hand, $\neg A$ is equivalent to $A \downarrow A$ and to $A \mid A$; $(A \downarrow B) \downarrow (A \downarrow B)$ is equivalent to $A \lor B$ and $(A \mid B) \mid (A \mid B)$ is equivalent to $A \land B$. Since both $\{\neg, \lor\}$ and $\{\neg, \land\}$ are adequate, it follows that $\downarrow$ or $|$ is adequate.
A formal theory $T$ has four parts $(\mathcal{S}, \mathcal{F}, \mathcal{A}, \mathcal{R})$ where

1. $\mathcal{S}$: a countable set of symbols
   $x \in \mathcal{S}^*$, the set of all finite strings over $\mathcal{S}$, is called an expression.

2. $\mathcal{F}$: the set of well-formed formulas
   $\mathcal{F} \subseteq \mathcal{S}^*$ and whether $x \in \mathcal{F}$ can be effectively verified.

3. $\mathcal{A}$: the set of axioms; $\mathcal{A} \subseteq \mathcal{F}$ and its membership problem can be effectively verified.

4. $\mathcal{R}$: rules of inference
   $\mathcal{R} = \{R_1, R_2, \ldots, R_n\}$ where each $R_i$ is a $k_i + 1$-ary relation over $\mathcal{F}$ written as
   \[
   \frac{A_1, A_2, \ldots, A_{k_i}}{A_{k_i + 1}}.
   \]
   $R_i$ can be effectively verified when given $A_1, \ldots, A_{k_i + 1}$. The lower term $A_{k_i + 1}$ is called a direct consequence.
Definition (Proof)
A proof in $\mathcal{T}$ is a sequence
\[ A_1, \ldots, A_n \]
of wfs such that, for each $i$, either $A_i \in \mathcal{A}$ or is a direct consequence of the preceding wfs by using one of the rules of inference.

Definition (Theorem)
A theorem of $\mathcal{T}$ is a wf $\mathcal{A}$ of $\mathcal{T}$ such that there is a proof where the last wf is $\mathcal{A}$. 
Definition (Decidability)

A theory $\mathcal{T}$ is called **decidable** if there is an algorithm for determining, given any wf $A$, whether there is a proof of $A$. Otherwise, if no such an algorithm does exist, it is called **undecidable**.
Definition (Consequence)

A wf $A$ is a consequence in $T$ of a set $\Gamma$ of wfs iff there is a sequence $A_1, \ldots, A_n$ of wfs such that

1. $A_n = A$;
2. for each $i$, either $A_i$ is an axiom, or $A_i \in \Gamma$, or $A_i$ is a direct consequence of the preceding wfs.

It is written as $\Gamma \vdash A$.

Remarks

1. When $\Gamma$ is a finite set $\{B_1, \ldots, B_k\}$, we write $B_1, \ldots, B_k \vdash A$ instead of $\{B_1, \ldots, B_k\} \vdash A$.
2. $\vdash A$ means $A$ is a theorem of $T$. 
Observations

1. If $\Gamma \subseteq \Delta$ and $\Gamma \vdash A$, then $\Delta \vdash A$.
2. $\Gamma \vdash A$ if and only if there is a finite subset $\Delta \subseteq \Delta$ such that $\Delta \vdash A$.
3. If $\Delta \vdash A$ and for each $B \in \Delta$ we have $\Gamma \vdash B$, then $\Gamma \vdash A$. 
Axiomatic Theory for the Propositional Calculus

The formal theory $L = (\mathcal{S}, \mathcal{F}, \mathcal{A}, \mathcal{R})$ is as follows.

1. $\mathcal{S} = \{\neg, \Rightarrow, (, )\} \cup \{A_i | i \in \mathbb{N}\}$.  
   $\neg$ and $\Rightarrow$ are called **primitive connectives**; $A_i$ are called **statement letters**.

2. The set of wfs $\mathcal{F}$ is defined recursively as
   - all statement letters are wfs;
   - if $A$ and $B$ are wfs, so are $(\neg A)$ and $(A \Rightarrow B)$.

3. Let $A$, $B$, and $C$ be any wfs of $L$. $\mathcal{A}$ contains
   - (A1) $(A \Rightarrow (B \Rightarrow A))$.
   - (A2) $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$.
   - (A3) $(((\neg B) \Rightarrow (\neg A)) \Rightarrow (((\neg B) \Rightarrow A) \Rightarrow B))$.

4. The only inference rule is **modus ponens**

$$
\begin{array}{c}
A, (A \Rightarrow B) \\
\hline
B
\end{array} \quad (MP)
$$
Remarks

(D1) \((A \land B) \text{ for } \neg(A \Rightarrow \neg B)\)

(D2) \((A \lor B) \text{ for } (\neg A) \Rightarrow B\)

(D3) \((A \iff B) \text{ for } (A \Rightarrow B) \land (B \Rightarrow A)\)

Lemma

\[\vdash_L A \Rightarrow A \text{ for all wfs } A.\]

Proof.

1. \((A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A))\) (Axiom (A2))

2. \(A \Rightarrow ((A \Rightarrow A) \Rightarrow A)\) (Axiom (A1))

3. \((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)\) (from 1 and 2 by MP)

4. \(A \Rightarrow (A \Rightarrow A)\) (Axiom (A1))

5. \(A \Rightarrow A\) (from 3 and 4 by MP)
Proposition (Deduction Theorem)

If $\Gamma, A \vdash B$, then $\Gamma \vdash A \Rightarrow B$. In particular, if $A \vdash B$, then $\vdash A \Rightarrow B$.

Let $B_1, \ldots, B_n$ be a proof of $B$ from $\Gamma \cup \{A\}$ where $B_n = B$. We will show, by induction, $\Gamma \vdash A \Rightarrow B_i$ for $1 \leq i \leq n$.

1. **(Basis)** $B_1$ must be either in $\Gamma$ or an axiom of $L$ or $A$ itself.
   - $B_1 \in \Gamma$ or $B_1$ is an axiom: $B_1, B_1 \Rightarrow A \Rightarrow B_1, A \Rightarrow B_1$ is a proof.
   - $B_1 = A$: $A \Rightarrow A$ is proved on Page 25.

2. **(Induction)** $B_i$ is either in $\Gamma$ or an axiom of $L$ or $A$ itself or by MP for $1 < i \leq n$.
   - The first three cases are the same as for $i = 1$.
   - $B_i$ is the direct consequence of $B_j$ and $B_m = B_j \Rightarrow B_i$ by MP where $j < i$ and $m < i$. By induction, we have $\Gamma \vdash A \Rightarrow B_j$ and $\Gamma \vdash A \Rightarrow (B_j \Rightarrow B_i)$. By Axiom (A2), we have $\vdash (A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))$. By twice applications of MP, we get a proof of $A \Rightarrow B_i$. 
Corollaries

- $A \Rightarrow B, B \Rightarrow C \vdash A \Rightarrow C$
- $A \Rightarrow (B \Rightarrow C), B \vdash A \Rightarrow C$
Lemmas I

For any wfs $A$ and $B$, the followings are theorems of $L$.

(a) $\neg\neg B \Rightarrow B$

Proof.

1. $(\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B)$ (Axiom schema (A3))
2. $\neg B \Rightarrow \neg B$ (the lemma on Page 25)
3. $(\neg B \Rightarrow \neg\neg B) \Rightarrow B$ (1, 2, and the corollary on Page 27)
4. $\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B)$ (Axiom (A1))
5. $\neg\neg B \Rightarrow B$ (3, 4, the corollary on Page 27)
(b) \( B \Rightarrow \neg \neg B \)

Proof.

1. \((\neg \neg \neg B \Rightarrow \neg B) \Rightarrow ((\neg \neg \neg B \Rightarrow B) \Rightarrow \neg \neg B)\) (Axiom (A3))
2. \(\neg \neg \neg B \Rightarrow \neg B\) (Part (a))
3. \((\neg \neg \neg B \Rightarrow B) \Rightarrow \neg \neg B\) (1, 2, MP)
4. \(B \Rightarrow (\neg \neg \neg B \Rightarrow B)\) (Axiom (A1))
5. \(B \Rightarrow \neg \neg B\) (3, 4, the corollary on Page 27)
(c) \( \neg A \Rightarrow (A \Rightarrow B) \)

1. \( \neg A \) (Hyp)
2. \( A \) (Hyp)
3. \( A \Rightarrow (\neg B \Rightarrow A) \) (Axiom (A1))
4. \( \neg A \Rightarrow (\neg B \Rightarrow \neg A) \) (Axiom (A1))
5. \( \neg B \Rightarrow A \) (2, 3, MP)
6. \( \neg B \Rightarrow \neg A \) (1, 4, MP)
7. \( (\neg B \Rightarrow \neg A) \Rightarrow (((\neg B \Rightarrow A) \Rightarrow B) \) (Axiom (A3))
8. \( (\neg B \Rightarrow A) \Rightarrow B \) (6, 7, MP)
9. \( B \) (MP)
10. \( \neg A, A \vdash B \) (1–9)
11. \( \neg A \vdash A \Rightarrow B \) (10, Deduction Theorem)
12. \( \vdash \neg A \Rightarrow (A \Rightarrow B) \) (11, Deduction Theorem)
Lemmas IV

(d) \((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)\)

Proof.

1. \(\neg B \Rightarrow \neg A\) (Hyp)
2. \((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)\) (Axiom (A3))
3. \(A \Rightarrow (\neg B \Rightarrow A)\) (Axiom (A1))
4. \((\neg B \Rightarrow A) \Rightarrow B\) (1, 2, MP)
5. \(A \Rightarrow B\) (3, 4, the corollary on Page 27)
6. \(\neg B \Rightarrow \neg A \vdash A \Rightarrow B\) (1–5)
7. \(\vdash (\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)\) (6, Deduction Theorem)
Lemmas V

\((e)\) \(\vdash (A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)\)

1. \(A \Rightarrow B\) (Hyp)
2. \(\neg\neg A \Rightarrow A\) (Part(a))
3. \(\neg\neg A \Rightarrow B\) (1, 2, the corollary on Page 27)
4. \(B \Rightarrow \neg\neg B\) (Part(b))
5. \(\neg\neg A \Rightarrow \neg\neg B\) (3, 4, the corollary on Page 27)
6. \((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A)\) (Part(d))
7. \((\neg B \Rightarrow \neg A)\) (5, 6, MP)
8. \(A \Rightarrow B \vdash (\neg B \Rightarrow \neg A)\) (1–7)
9. \(\vdash (A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)\) (8, Deduction Theorem)
Lemmas VI

(f) \( \vdash A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)) \)

Proof.

1. \( A, A \Rightarrow B \vdash B \) (MP)
2. \( \vdash A \Rightarrow ((A \Rightarrow B) \Rightarrow B) \) (Deduction Theorem)
3. \( \vdash ((A \Rightarrow B) \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)) \) (Part (e))
4. \( \vdash A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)) \) (the corollary on Page 27)
(g) $\vdash (A \Rightarrow B) \Rightarrow (((\neg A \Rightarrow B) \Rightarrow B)$

1. $A \Rightarrow B$ (Hyp)
2. $\neg A \Rightarrow B$ (Hyp)
3. $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$ (Part(e))
4. $\neg B \Rightarrow \neg A$ (1, 3, MP)
5. $(\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$ (Part (e))
6. $\neg B \Rightarrow \neg A$ (2, 5, MP)
7. $(\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)$ (Axiom (A3))
8. $(\neg B \Rightarrow \neg A) \Rightarrow B$ (6, 7, MP)
9. $B$ (4, 8, MP)
10. $\vdash (A \Rightarrow B) \Rightarrow (((\neg A \Rightarrow B) \Rightarrow B)$ (Deduction Theorem)
Soundness Theorem

Proposition (Soundness)

Every theorem of $L$ is a tautology.

Proof.

1. All axioms of $L$ are tautologies.
2. The modus ponens of two tautologies is again a tautology.
Lemma

Let $A$ be a wf, and let $B_1, \ldots, B_k$ be the statement letters in $A$. For any assignment of truth values to $B_1, \ldots, B_k$, define

- $B'_i = B_i$ if $B_i$ takes the value $T$; otherwise $B'_i = \neg B_i$ if $B_i$ takes the value $F$;
- $A' = A$ if $A$ takes the value $T$; otherwise $A' = \neg A$ if $A$ takes the value $F$.

Then

$$B'_1, \ldots, B'_k \vdash A'. \quad (1)$$

Example

Let $A$ be $\neg(\neg A_2 \Rightarrow A_5)$. We have

$$A_2, \neg A_5 \vdash \neg\neg(\neg A_2 \Rightarrow A_5). \quad (2)$$
We show this by induction on the structure of $A$.

**(Basis)** $A$ is a statement letter $B_1$. It reduces to show that $B_1 \vdash B_1$ and $\neg B_1 \vdash \neg B_1$, which are proved on Page 25.

**(Induction)**

1. $A$ is $\neg B$:
   - $A$ takes the value $F$ and $B$ takes the value $T$. By induction, we have $B'_1, \ldots, B'_{k} \vdash B$. By the lemma on Page 29, we have $\vdash B \Rightarrow \neg\neg B$. Thus, $B'_1, \ldots, B'_{k} \vdash \neg A$.
   - $A$ takes the value $T$ and $B$ takes the value $F$. By induction, we have $B'_1, \ldots, B'_{k} \vdash \neg B$, which is equal to $B'_1, \ldots, B'_{k} \vdash A$.

2. $A$ is $B \Rightarrow C$:
   - $B$ takes $F$. We have $B'_1, \ldots, B'_{k} \vdash \neg B$. By the lemma on Page 30 ($\vdash \neg B \Rightarrow B \Rightarrow C$), we have $B'_1, \ldots, B'_{k} \vdash B \Rightarrow C$, and $B \Rightarrow C$ is $A'$.
   - $C$ takes $T$. We have $B'_1, \ldots, B'_{k} \vdash C$. Then, by axiom $(A1)$ $(C \Rightarrow B \Rightarrow C)$, $B'_1, \ldots, B'_{k} \vdash B \Rightarrow C$, and $B \Rightarrow C$ is $A'$.
   - $B$ takes $T$ and $C$ takes $F$. We have $B'_1, \ldots, B'_{k} \vdash B$ and $B'_1, \ldots, B'_{k} \vdash \neg C$. Then by the lemma on Page 33 ($B \Rightarrow (\neg C \Rightarrow \neg(B \Rightarrow C))$), we have $B'_1, \ldots, B'_{k} \vdash \neg(B \Rightarrow C)$, and $\neg(B \Rightarrow C)$ is $A'$. 
Completeness Theorem

Proposition (Completeness Theorem)

If a wf $A$ of $L$ is a tautology, then it is a theorem of $L$. That is, $A$ is a tautology implies $\vdash_L A$.

Let $B_1, \ldots, B_k$ be statement letters in $A$. Then

$$B'_1, \ldots, B'_{k-1}, B_k \vdash A ;$$
$$B'_1, \ldots, B'_{k-1}, \neg B_k \vdash A .$$

Hence

$$B'_1, \ldots, B'_{k-1} \vdash B_k \Rightarrow A ;$$
$$B'_1, \ldots, B'_{k-1} \vdash \neg B_k \Rightarrow A .$$

Then by the lemma on Page 34, we have $B'_1, \ldots, B'_{k-1} \vdash A$. This process can be continued, and finally we will get $\vdash A$. 