Fundamentals of Mathematics
Lecture 5: Induction

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Mathematical Induction: Basic Form I

$$\frac{1 \in F;\ \forall_n [n \in F \implies n+1 \in F]}{\mathbb{N} \subseteq F}$$

or

$$\frac{P(1);\ \forall_{n\in\mathbb{N}} [P(n) \implies P(n+1)]}{\forall_{n\in\mathbb{N}} P(n)}.$$

Mathematical Induction: Basic Form II

Example

1. Show that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.
2. Show that $1^3 + 2^3 + \cdots + n^3 = [\frac{n(n+1)}{2}]^2$.
3. Show that $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

Mathematical Induction: Basic Form III

Example

How many regions are there that n lines in a plane can divide at most? (2D $\rightarrow$ 3D)

Example (Wrong)

Show that all numbers are equal.

Mathematical Induction: Basic Form IV

Proof.

Let A be a set of n numbers.

Basis : $n = 1$. The claim follows for just one number.

Induction : Suppose "Given any set of k numbers where $k < n$, these numbers must be equal."

1. Let A be a set of n numbers and let $\alpha \in A$. Then all numbers in $A \backslash \{\alpha\}$ are equal by induction hypothesis.
2. Let $\beta \in A \backslash \{\alpha\}$. We will show that α equals to β, and hence all numbers in A are equal.
3. Remove a number from A and keep α and β inside. Hence all numbers in this set must be equal, and thus α equals to β.

$\square$

Mathematical Induction: Strong Form

$$\frac{P(1);\ \forall_{n\in\mathbb{N}}[\forall_{1\leq k<n}P(k) \implies P(n)]}{\forall_{n\in\mathbb{N}}P(n)}.$$

1. Basis: Show that $P(1)$ is true.
2. Induction: Assume $P(1), P(2), \ldots, P(k)$ are true, and derive that $P(k+1)$ is true for all $k \geq 1$.

Examples I

1. Show that $2^n \geq n^2$ for $n \geq 4$.
2. The marble game of two piles.
3. Let $r(n)$ be the number of solutions (x, y) for $x + 2y = n$, where $n, x, y \geq 0$ and are integers. Show that $r(n) = \frac{n+1}{2} + \frac{1+(-1)^n}{4}$.

Examples II

4. Let $a_{2k} = 3k^2$, $a_{2k-1} = 3k(k-1)+1$. Let $S_n = a_1 + a_2 + \cdots + a_n$. Show that

$$\begin{cases} S_{2k-1} = \frac{k}{2}(4k^2 - 3k + 1) \\ S_{2k} = \frac{k}{2}(4k^2 + 3k + 1) \qquad \text{for } k \geq 1. \end{cases}$$

Examples III

5. Show that

$$\frac{a_1 + \cdots + a_n}{n} \geq (a_1 \cdots a_n)^{\frac{1}{n}} \text{ where } a_i \geq 0$$

Examples IV

6. Show that for any $n + 1$ numbers out of $1, 2, \ldots, 2n$, there always exist two numbers such that one is a multiple of the other.

Examples V

7. Let $S_n = 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!$. Show that $S_n = (n+1)! - 1$.
8. Show that $(1+a)^n \geq 1 + na$ for any integer $n \geq 0$ and real number $a > -1$.

Examples VI

9. Show that
$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$$
for all integers $n \geq 1$.

Examples VII

10. Prove the Euler's formula for planar graphs: $E = V + F - 1$ for connected planar graphs.

Examples VIII

11. Let G be any graph. Show that there exists an independent set I of G such that all nodes in G is reachable from I in at most two steps.

Noetherian Induction

- Well-founded ordering
- Induction principle

Well-Founded Ordering

- We say a strict partial order $(A, \prec)$ is well-founded if there is no infinite descending chain

$$a_0 \succ a_1 \succ \cdots \succ a_k \succ \cdots$$

where $a_k \in A$.

Strict Partial Order

1. irreflexive: $a \not\prec a$ for all $a \in A$
2. transitive: $a \prec b$ and $b \prec c \implies a \prec c$.

Examples

- $(\mathbb{N}, <)$
- Set inclusion: $A \prec B$ iff $A \subsetneq B$
- Interval overlap relation
- string containment relation: $x \prec y$ iff x occurs in y

Examples

- $(\mathbb{N}, <)$ is well-founded.
- $(\mathbb{Z}, <)$ is not well-founded.
- "Set inclusion relation" is well-founded iff the universe is a finite set.
- Most finite structures are well-founded.

Noetherian Induction Principle

Let $(A, \prec)$ be well-founded. Then

$$\frac{\forall_{x \in A}[\forall_{y \in A, y \prec x} P(y) \implies P(x)]}{\forall_{x \in A} P(x)}$$

Remark

- *Where is the basis? They are the minimal elements of $(A, \prec)$.*
- *This is a backward process (i.e., from x towards the bases).*

More on Well-Founded Orderings

- Structural ordering (Inductive ordering)
- Lexicographic ordering
- Multiset ordering

They can be used to construct new well-founded orderings based on known ones.

Structural Ordering

Tree

The definition of a tree can be

- Basis: A node called root is a tree.
- Induction: If $T_1, \ldots, T_k$ are trees, then $N(T_1, T_2, \ldots, T_k)$ is a tree where N is a node.

Expression

The definition of an expression is as follows.

- Basis: Any number or variable is an expression
- Induction: If E and F are expressions, then so are

$$E + F, E * F, \text{ and } (E).$$

Recursive Definition for Structures

It has the following pattern in order to define a structure $\mathcal{S}$:

- Basis: A set of basic elements (i.e., the basis) are assigned to $\mathcal{S}$.
- Induction: New elements of $\mathcal{S}$ are constructed based on a finite number of elements in $\mathcal{S}$ recursively.

Remark

- *When the basis is well-founded, the above inductive definition induces a well-founded ordering.*
- *Induction based on this is called the structural induction.*
- *Structural induction is especially important to computer science since one of our jobs is to deal with recursive structures (e.g., trees, graphs, lists, strings).*

Lexicographic Ordering

Let $(A, \prec)$ be a strict partial order.

- $(A \times A, \prec_{\text{lex}})$: Define $(a_1, a_2) \prec_{\text{lex}} (b_1, b_2)$ iff $a_1 \prec b_1$ or ($a_1 = b_1$ and $a_2 \prec b_2$).
- $(A^k, \prec_{\text{lex}})$: Define $(a_1 a_2 \ldots a_k) \prec_{\text{lex}} (b_1 b_2 \ldots b_k)$ iff there exists t where $1 \leq t \leq k$ such that $a_i = b_i$ for i with $1 \leq i < t$ and $a_t \prec b_t$.

Theorem

If $(A, \prec)$ is well-founded, then $(A^k, \prec_{\text{lex}})$ is well-founded.

Example

$(1, 5) <_{\text{lex}} (2, 2)$

Multiset Ordering

Let $(A, \prec)$ be a strict partial order. Let $M(x)$ be the multiplicity of x in a multiset M.

- $M \prec_{\mathrm{mul}} N$ iff $M(x) > N(x)$ implies there is $y \succ x$ such that $M(y) < N(y)$.

Theorem (Dershowitz & Manna, 1979)

If $(A, \prec)$ is well-founded, then $(A, \prec_{\mathrm{mul}})$ is well-founded.

Example

$\{1, 1, 1, 4, 4\} <_{\mathrm{mul}} \{1, 2, 4, 4\}$

Summary

- Mathematical Induction principle: basic form & strong form
- Noetherian induction principle
- Well-founded ordering
- Structural induction principle

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