Fundamentals of Mathematics
Lecture 4: Mathematical Statements

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Axiom, Postulate, Assumption I

Basic assumptions or principles for a system or theory.

Definition

Let T be a theory. We say Δ, a set of true facts in T, is an axiom system for T iff

$$T \models \psi \text{ if and only if } \Delta \models \psi$$

for all propositions ψ about T.

- sound: $\Delta \models \psi \implies T \models \psi$;
- consistent: no contradiction;
- complete: $T \models \psi \implies \Delta \models \psi$.

And it should be independent. That is, there is no $\Delta' \subsetneq \Delta$ such that $\Delta' \models \Delta$.

Axiom, Postulate, Assumption II

Example

1. Euclidean geometry
2. natural numbers: (Peano's Axioms)
   1. $0 \in \mathbb{N}$.
   2. $\forall_{n \in \mathbb{N}} S(n) \in \mathbb{N}$.
   3. $\forall_{n \in \mathbb{N}} S(n) \neq 0$.
   4. $(\forall_{m,n \in \mathbb{N}}) S(m) = S(n) \implies m = n$.
   5. Mathematical Induction
3. vector space
4. probability
5. group theory

Algorithm, Procedure, Function

Every one should know their distinctions in programming language.

Theorem, Lemma, Proposition, Corollary

- proposition: a fact in a theory
- lemma: an important or useful fact that is used to prove a theorem in a theory
- theorem: a major result in a theory
- corollary: a direct consequence of a theorem or proposition

Example

1. Fundamental Theorem of Calculus
2. Fundamental Theorem of Arithmetic
3. Fundamental Theorem of Algebra

Conjecture

An observation that is possibly true in a theory

- A list of conjectures

Definition I

Define the meaning of a term in a theory. It often reflects some concept in this theory. (Similar to "declaration" in programming language.)

- precise (+3)
- accurate (+3)
- clear (+2)
- concise (+1)

+3: must; +2: should; +1: would; 0: neutral;

Definition II

Why should we have to define the meaning of a term?

- To avoid ambiguity.
- To introduce important concept or idea in a theory.
- To define constructs in a theory.
- To shorten the description of a theory.

$\Longrightarrow$ It is the foundation of a theory.

Hence one should

1. follow the definitions strictly;
2. never define a term that is never used.

Principle

Fundamental rule or law that governs a system.

Proof

- correct $(+3)$:
  1. sound: Each derivation must be sound.
  2. complete: The whole derivation actually asserts the claim.
- precise $(+3)$
- clear $(+2)$:
  1. Does the idea expose clearly?
  2. What are the techniques that you apply?
  3. Is the style appropriate?
- concise $(+1)$
- elegant $(+1)$

formal and informal proofs (homework exercises)

Example I

A good example can help the reader capture the essence of the discussion quickly.

- Keep it simple, as simple as possible, but not simpler.
- It has to be general enough so that it can reflect the point.

Example II

Example

Definition: A period of a sting x is an integer p, where $0 < p \leq |x|$, such that

$$x[i] = x[i+p] \text{ for all } 1 \leq i \leq |x| - p.$$

Bad examples:

1. Let $x = a$ and $p = 1$.
2. Let $x = aa$ and $p = 2$.
3. Let $x = aa$ and $p = 1$.
4. Let $x = abab$ and $p = 2$.

Good example: Let $x = ababa$ and $p = 4$.

Exercise

D. E. Knuth rated the exercises in his books from 0 to 50.

- 0: immediate
- 10: simple (one minute)
- 20: medium (quarter hour)
- 30: moderately hard (more than two hours; even more time when the TV is on)
- 40: term project
- 50: research problem

In reality, an exercise could be

1. something that is very important so that a student is supposed to have to know;
2. something that is tedious so that the teacher doesn't want to teach.

Correctness

In order to make a proof or an algorithm correct, the following ingredients are the most important.

- soundness
- completeness

Some Common Usages

- Without loss of generality, ...
- ... if and only if ...
- necessary condition
- sufficient condition
- $P(n)$ is true for all but finite elements.

References I

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- Wikipedia, http://wikipedia.org/.
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