Concrete Mathematics

Midterm Examination

Nov. 15, 2010

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Problem 1 (15 points) Evaluate $\sum_{k=1}^{n} k^2 H_k$ in the closed form.

Problem 2 (15 points) Evaluate $\sum_{k=1}^{n} k^4$ in the closed form.

Problem 3 (15 points) Let $\Delta^1 f(x) = \Delta f(x) = f(x+1) - f(x)$ and $\Delta^m f(x) = \Delta(\Delta^{m-1} f(x))$ for integers $m \geq 2$. Show that $\Delta^n x^n = n!$ for all integers $n \geq 1$.

Problem 4 (15 points) Prove that

$$\Delta \sin x = 2 \sin(\frac{1}{2}) \cos(x + \frac{1}{2})$$

and

$$\Delta \cos x = -2 \sin(\frac{1}{2}) \sin(x + \frac{1}{2}) \ .$$

Use the above results to show that

$$\sum_{k=1}^{n} \sin k = \frac{\sin(\frac{n+1}{2}) \sin(\frac{n}{2})}{\sin(\frac{1}{2})} \ .$$

Problem 5 (15 points) Let $\Delta^1 f(x) = \Delta f(x) = f(x+1) - f(x)$ and $\Delta^m f(x) = \Delta(\Delta^{m-1} f(x))$ for integers $m \geq 2$. Show that $f(x+3) = f(x) + 3\Delta f(x) + 3\Delta^2 f(x) + \Delta^3 f(x)$.

Problem 6 (15 points) Show that if $2^n + 1$ is prime then n is a power of 2.

Problem 7 (15 points) Prove or disprove: $\lceil x \rceil + \lceil y \rceil + \lceil x + y \rceil \leq \lceil 2x \rceil + \lceil 2y \rceil$.