# Concrete Mathematics

Midterm Exam

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**Problem 1** Evaluate $\sum_{k=1}^{n} k^3$ in closed form.

**Problem 2** Show that $(x+y)^{\underline{4}} = x^{\underline{0}}y^{\underline{4}} + 4x^{\underline{1}}y^{\underline{3}} + 6x^{\underline{2}}y^{\underline{2}} + 4x^{\underline{3}}y^{\underline{1}} + x^{\underline{4}}y^{\underline{0}}$.

**Problem 3** Prove the identity

$$\left(\sum_{k=1}^{n} a_k x_k\right)\left(\sum_{k=1}^{n} b_k y_k\right) = \left(\sum_{k=1}^{n} a_k y_k\right)\left(\sum_{k=1}^{n} b_k x_k\right) + \sum_{1\le j<k\le n} (a_j b_k - a_k b_j)(x_j y_k - x_k y_j).$$

**Problem 4** Evaluate $\sum_{k=1}^{n} k^{\underline{2}} H_k$ in closed form.

**Problem 5** Prove that $\Delta \cos x = -2\sin(\frac{1}{2})\sin(x+\frac{1}{2})$. Use the above result to show that $\sum_{k=1}^{n} \sin k = \frac{\sin(\frac{n+1}{2})\sin(\frac{n}{2})}{\sin(\frac{1}{2})}$ by finite calculus.

**Problem 6** Prove or disprove: $\lfloor x\rfloor + \lfloor y\rfloor + \lfloor x+y\rfloor \le \lfloor 2x\rfloor + \lfloor 2y\rfloor$.

**Problem 7** Let $K_0 = 1$ and $K_{n+1} = 1 + \min(2K_{\lfloor n/2\rfloor}, 3K_{\lfloor n/3\rfloor})$ for $n \ge 0$. Prove or disprove that $K_n \ge n$.

**Problem 8** Let $Orb(n,m) = \{nk \mod m \mid 0 \le k < m\}$ where $m$ and $n$ are positive integers. What is the cardinality (or size) of $Orb(n,m)$? Prove your claim.