Theory of Computation
Chapter 7

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Parameters for a Complexity Class

- model of computation: multi-string Turing machine
- modes of computation
  1. deterministic mode
  2. nondeterministic mode
- a fixed resource we wish to bound
  1. time
  2. space
- a bound f mapping from $\mathbb{N}$ to $\mathbb{N}$.

Definition 7.1: Proper Function

$f : \mathbb{N} \rightarrow \mathbb{N}$ is proper if

1. f is non-decreasing (i.e., $f(n+1) \geq f(n)$);
2. there is a k-string TM M_f with I/O such that for any input x of length n, M_f computes $\sqcup^{f(n)}$ in time $O(n + f(n))$.

Example 7.1

1. $f(n) = c$ is proper.
2. $f(n) = \lceil \lg n \rceil$ is proper.
   Since $\lfloor \lg n \rfloor + 1$ is the length of binary digits for n.
3. $(\lg n)^2, n \lg n, n^2, n^3 + 3n, 2^n, \sqrt{n}, n!$ are all proper.

Remark If f and g are proper, then so are $f + g$, $f \cdot g$, 2^g, and $f \circ g$. ($f(n) \geq n$ for the last case)

Definition

A Turing machine M (with or without I/O, deterministic or nondeterministic) is precise if there are functions f and g such that for any input x, $M(x)$ halts after precisely $f(|x|)$ steps and uses precisely $g(|x|)$ spaces.

Proposition 7.1

Suppose that a TM M (deterministic or not) decides a language L within time (or space) $f(n)$ where f is a proper function. Then there is a precise TM M' that decides L in time (or space, resp.) $O(f(n))$.

1. Compute $M_f(x)$. Use the output of $M_f(x)$ as a "yardstick" (alarm clock).

2. Run M according to the "yardstick."

Definition: Complexity Classes

1. $\texttt{TIME}(f)$: deterministic time

   $\texttt{SPACE}(f)$: deterministic space

   $\texttt{NTIME}(f)$: nondeterministic time

   $\texttt{NSPACE}(f)$: nondeterministic space

   where f is always a proper function.

2. $\texttt{TIME}(n^k) = \bigcup_{j>0} \texttt{TIME}(n^j)$ $(= \mathcal{P})$

   $\texttt{NTIME}(n^k) = \bigcup_{j>0} \texttt{NTIME}(n^j)$ $(= \mathcal{NP})$

3. $\text{PSPACE} = \texttt{SPACE}(n^k)$

   $\text{NPSPACE} = \texttt{NSPACE}(n^k)$

   $\text{EXP} = \texttt{TIME}(2^{n^k})$

   $\mathcal{L} = \texttt{SPACE}(\lg n)$

   $\mathcal{NL} = \texttt{NSPACE}(\lg n)$

Complement of Nondeterministic Classes

non-deterministic computation:

$$\begin{cases} \text{accepts a string if one successful computation exists;} \\ \text{rejects a string if all computations fail.} \end{cases}$$

Definition

1. Let $L \subseteq \Sigma^*$ be a language. The complement of L is $\bar{L} = \Sigma^* - L$.
2. The complement of a decision problem A, usually called A-complement, is the decision problem whose answer is "yes" if the input is "no" in A, "no" if the input is "yes" in A.

Example

1. SAT-complement (or coSAT): Given a Boolean expression ϕ in conjunctive normal form, is it unsatisfiable?

Definition

For any complexity class $\mathcal{C}$, let $co\mathcal{C}$ be the class $\{\bar{L} | L \in \mathcal{C}\}$.

Corollary $\mathcal{C} = co\mathcal{C}$ if $\mathcal{C}$ is a deterministic time or space complexity class.

That is, all deterministic time and space complexity classes are closed under complement.

Remark

If is an important open problem whether nondeterministic time complexity classes are closed under complement.

Halting Problem with Time Bounds

Definition

$H_f = \{M; x \mid M$ accepts input x after at most $f(|x|)$ steps$\}$
where $f(n) \geq n$ is a proper complexity function.

Lemma 7.1 $H_f \in \mathtt{TIME}(f(n)^3)$.

$(H_f \in \mathtt{TIME}(f(n) \cdot \lg^2 f(n)))$

Lemma 7.2

$H_f \notin \texttt{TIME}(f\lfloor \frac{n}{2} \rfloor)$.

Proof: By contradiction. Suppose M_{H_f} decides H_f in time $f\lfloor \frac{n}{2} \rfloor$. Define $D_f(M)$ as

$$\text{if } M_{H_f}(M;M) = \text{“yes” then “no”, else “yes”.}$$

What is $D_f(D_f)$?

If $D_f(D_f)=$ “yes”, then $M_{H_f}(M_{D_f};M_{D_f}) =$ “no”,
“no” “yes”.

Contradiction!

The Time Hierarchy Theorem

Theorem 7.1

If $f(n) \geq n$ is a proper complexity function, then the class $\texttt{TIME}(f(n))$ is strictly contained within $\texttt{TIME}(f(2n+1)^3)$.

Remark

A stronger version suggests that

$$\texttt{TIME}(f(n)) \subsetneq \texttt{TIME}(f(n) \lg^2 f(n)).$$

Corollary $\mathcal{P}$ is a proper subset of EXP.

1. $\mathcal{P}$ is a subset of $\texttt{TIME}(2^n)$.
2. $\texttt{TIME}(2^n) \subsetneq \texttt{TIME}((2^{2n+1})^3)$ (Time Hierarchy Theorem)
   $\texttt{TIME}((2^{2n+1})^3) \subseteq \texttt{TIME}(2^{n^2}) \subseteq \text{EXP}$.

The Space Hierarchy Theorem

If $f(n)$ is a proper function, then $\mathtt{SPACE}(f(n))$ is a proper subset of $\mathtt{SPACE}(f(n) \lg f(n))$.

(Note that the restriction $f(n) \geq n$ is removed from the Time Hierarchy Theorem.)

The Gap Theorem

Theorem 7.3 There is a recursive function f from $\mathbb{N}_0$ to $\mathbb{N}_0$ such that $\texttt{TIME}(f(n)) = \texttt{TIME}(2^{f(n)})$.

The Reachability Method

Theorem 7.4 Suppose that $f(n)$ is a proper complexity function.

1. $\texttt{SPACE}(f(n)) \subseteq \texttt{NSPACE}(f(n))$, $\texttt{TIME}(f(n)) \subseteq \texttt{NTIME}(f(n))$.
   ($\because$ DTM is a special NTM.)
2. $\texttt{NTIME}(f(n)) \subseteq \texttt{SPACE}(f(n))$.
3. $\texttt{NSPACE}(f(n)) \subseteq \texttt{TIME}(k^{\lg n + f(n)})$ for $k > 1$.

Corollary

$$\mathcal{L} \subseteq \mathcal{NL} \subseteq \mathcal{P} \subseteq \mathcal{NP} \subseteq \text{PSPACE}.$$

However, $\mathcal{L} \subsetneq$ PSPACE. Hence at least one of the four inclusions is proper.

Theorem 7.5: (Savitch's Theorem)
REACHABILITY $\in \texttt{SPACE}(\lg^2 n)$.

Corollary

1. $\texttt{NSPACE}(f(n)) \subseteq \texttt{SPACE}(f(n)^2)$ for any proper complexity function $f(n) \geq \lg n$.
2. PSPACE = NPSPACE

Immerman-Szelepscényi Theorem

Theorem 7.6 If $f \geq \lg n$ is a proper complexity function, then $\texttt{NSPACE}(f(n)) = co\texttt{NSPACE}(f(n))$.

Corollary $\mathcal{NL} = co\mathcal{NL}$.